Graph unique-maximum and conflict-free colorings

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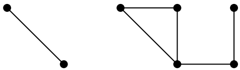
Graphs and vertex colorings

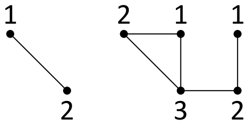
A *graph* $G = (V, E)$.

The vertex set: V.

The edge set E consists of subsets of V of cardinality 2.



(traditional) vertex coloring: an assignment of colors (i.e., positive integers) to the vertices such that no two adjacent vertices get the same color



Outline

- Introduction
  - Graphs and vertex colorings
  - Unique-maximum coloring and conflict-free coloring
  - Applications of these colorings
- Properties of UM and CF colorings
  - Separators and UM colorings
- Differences between CF and UM colorings
  - Graphs with different CF and UM chromatic numbers
  - Difference in computational complexity
- CF and UM colorings for grid graphs
- Conclusion, further work, open problems

Unique-maximum coloring

Def. A *unique-maximum coloring* of $G = (V, E)$ with k colors is a function $C\colon V \to \{1, \ldots, k\}$ such that for each simple path p in G the maximum color assigned to vertices of p occurs in *exactly one* vertex of p.

Given a graph G, the minimum k for which G has a unique-maximum coloring with k colors is called the *unique-maximum chromatic number* of G, denoted by $\chi_{\text{um}}(G)$.

In the literature, it is also known as:

vertex ranking
(Iyer, Ratliff, Vijayan, 1988)

ordered coloring
(Katchalski, McCuaig, Seager, 1995)

Motivation for unique-maximum coloring

parallel Cholesky decomposition of matrices (Liu, 1990)

planning efficient assembly of products in manufacturing systems (Iyer, Ratliff, Vijayan, 1988)

In general, the unique-maximum coloring problem can model situations where interrelated tasks have to be accomplished fast in parallel (assembly from parts, parallel query optimization in databases, etc.)

worst-case complexity of finding local optima in neighborhood structures (Llewellyn, Tovey, Trick, 1989)

Conflict-free coloring for paths of graphs

Def. A *conflict-free coloring* of $G = (V, E)$ with k colors is a function $C\colon V \to \{1, \ldots, k\}$ such that for each simple path p in G there is a vertex v in p, such that $C(v) \neq C(v')$, for every other vertex v' of p (i.e., $C(v)$ occurs uniquely in p).

Given a graph G, the minimum k for which G has a conflict-free coloring with k colors is called the *conflict-free chromatic number* of G, denoted by $\chi_{\text{cf}}(G)$.

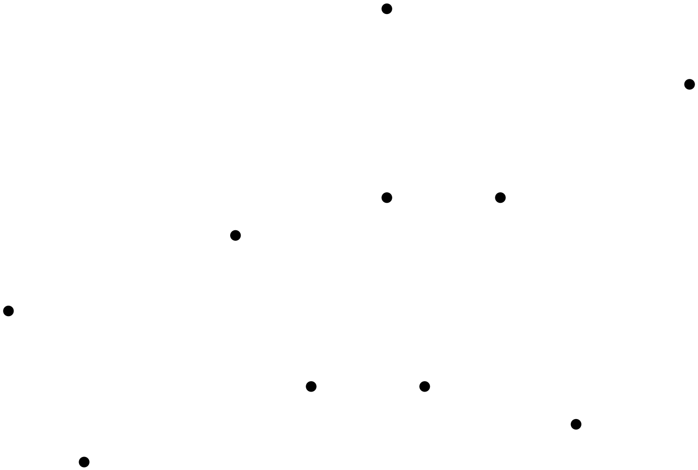
(Even, Lotker, Ron, Smorodinsky, 2003;
in a more general setting)

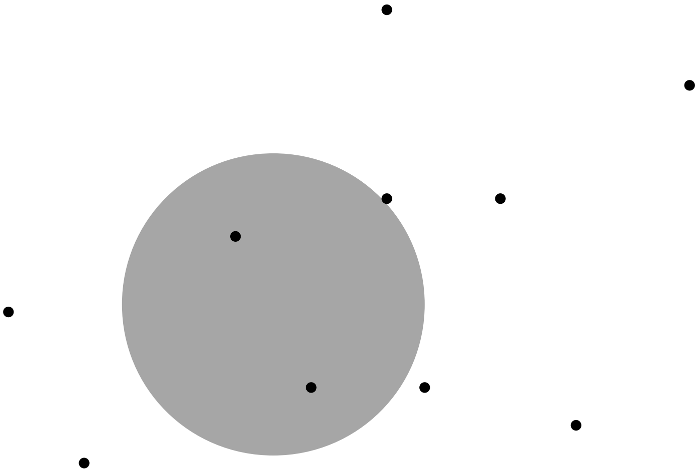
(CF colorings of a graph G) $\supseteq$ (UM colorings of a graph G)
and thus
$\chi_{\text{cf}}(G) \leq \chi_{\text{um}}(G)$

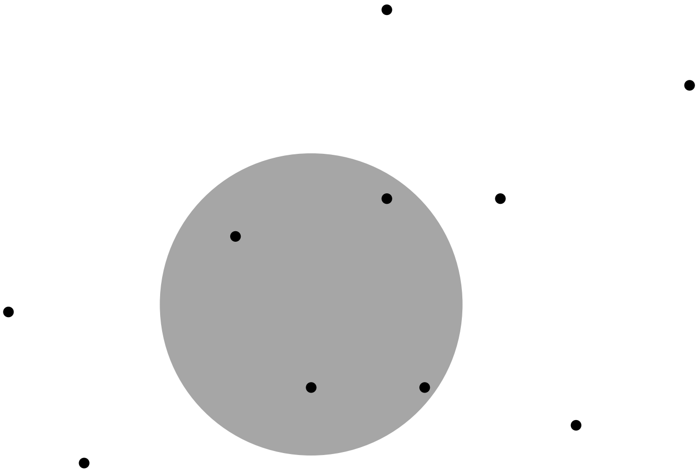
Motivation for conflict-free coloring

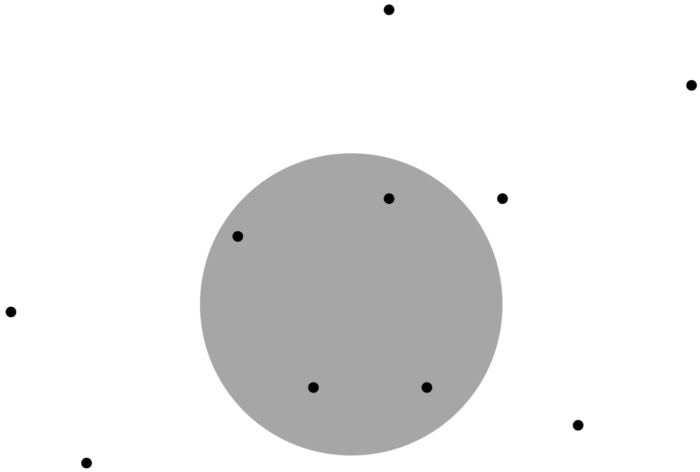
(Even, Lotker, Ron, Smorodinsky, 2003)

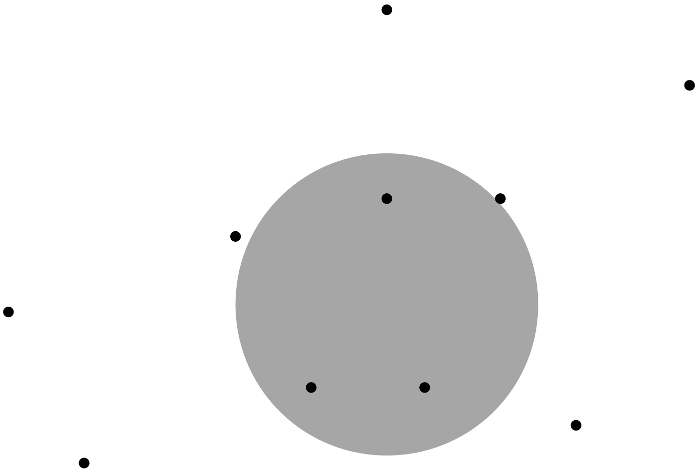
- Cellular networks consist of fixed position *base stations* (or antennas) that emit at a specific frequency, and *moving agents*.
- Each moving agent has a range of communication that can be modeled by a shape (like a disk). The range includes a subset S of the base stations. We want each such S to contain a base station with unique frequency in S.
- Model: base stations $\rightarrow$ points, frequencies $\rightarrow$ colors
- The frequency spectrum is expensive. Therefore, we try to minimize frequency use, i.e., reuse frequencies as much as possible.

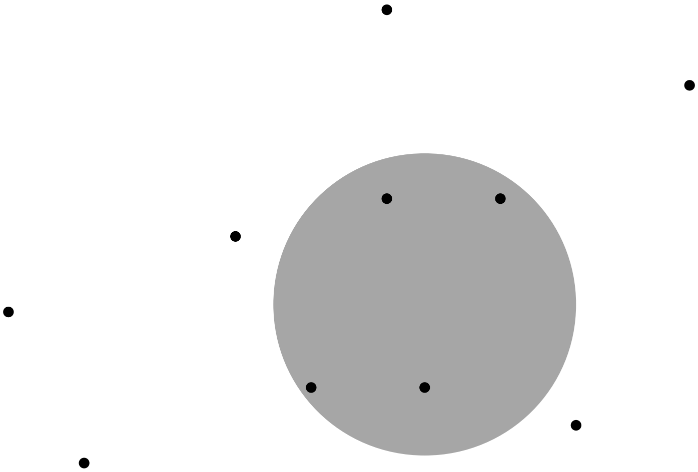


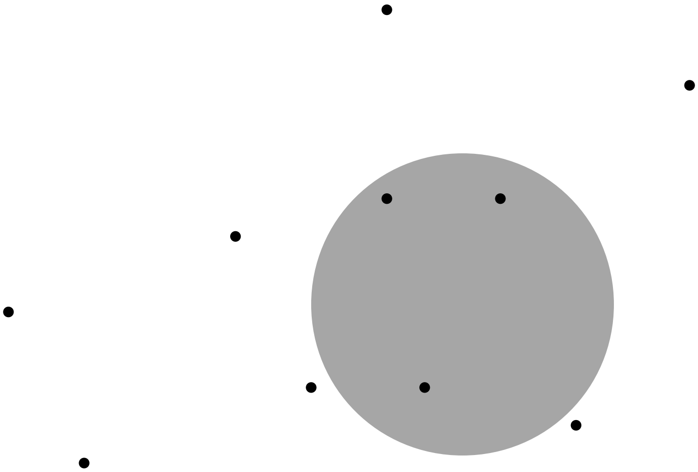


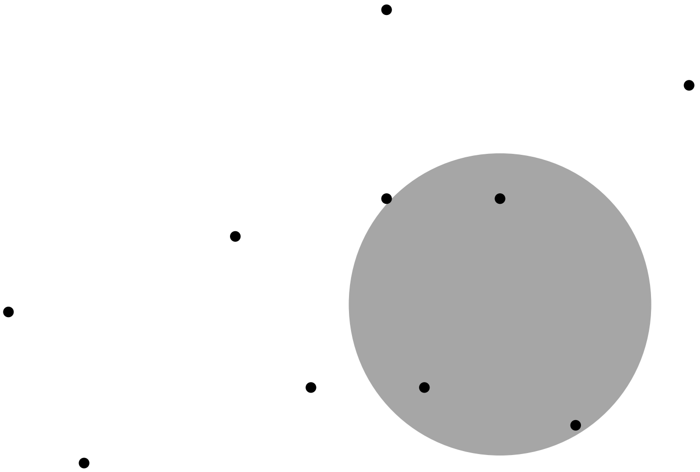


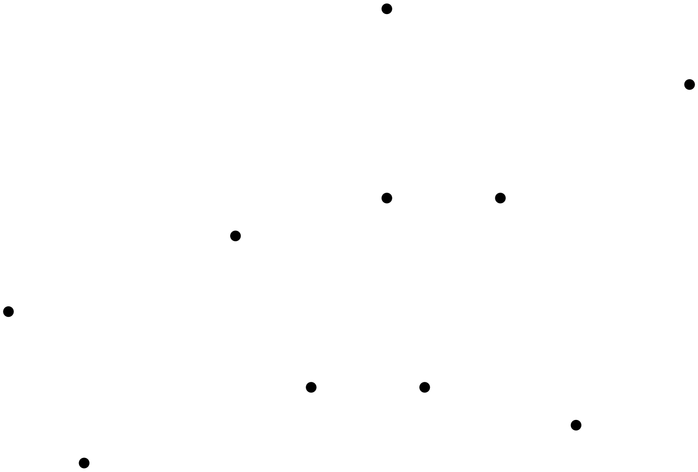


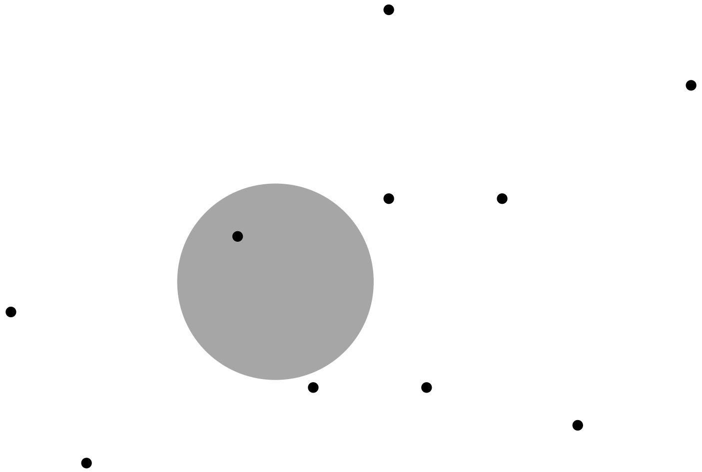


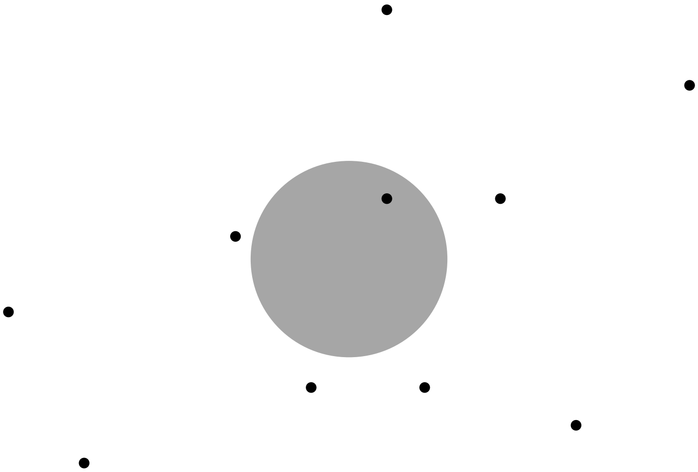


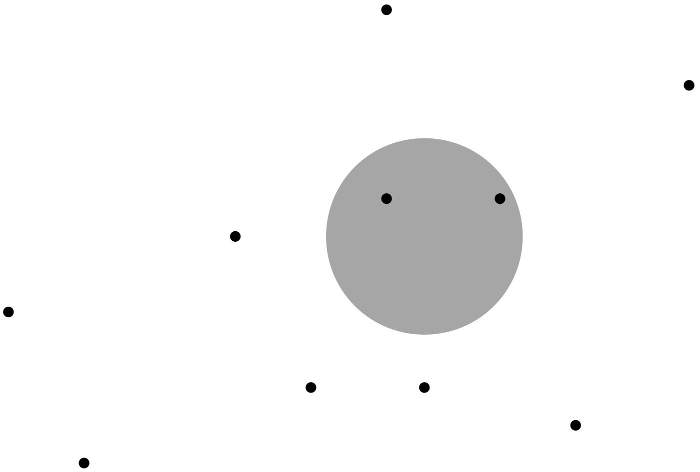












Properties – separators and UM colorings

Prop. If G is connected then in any UM coloring of G the maximum color occurs uniquely.

Def. Notation: $G - S$: deletion of vertices in $S \subseteq V$ from G

Def. A subset $S \subseteq V$ is a *separator* of a connected graph $G = (V, E)$ if $G - S$ is disconnected or empty. A separator S is *inclusion minimal* if no strict subset $S' \subset S$ is a separator.

Prop. Given a connected graph G and a UM coloring C of it, consider the set of vertices U which are colored with uniquely occurring colors in C. Then U is a separator of G.

Close relation between separators and UM colorings.

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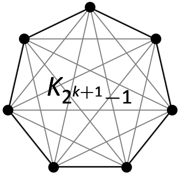

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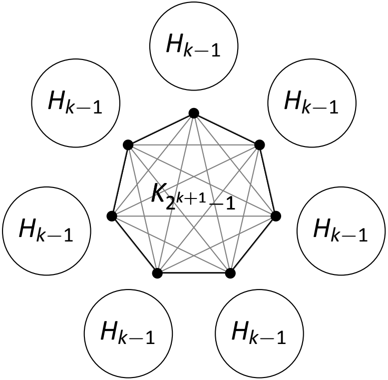

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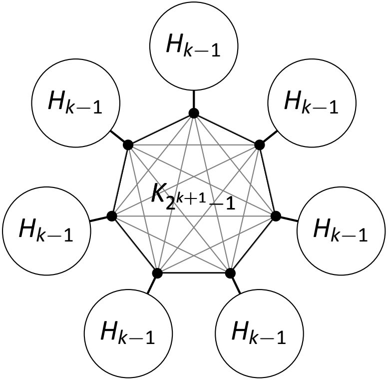
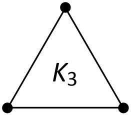
$2^{k+1} - 1$ copies of H_{k-1}

$2^{k+1} - 1$ edges



H_1



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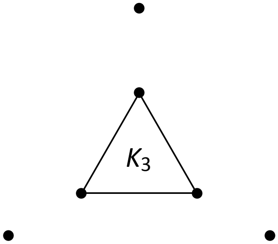

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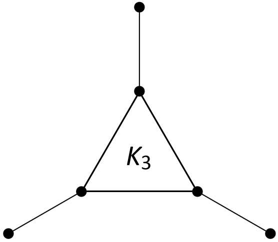
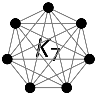


H_2



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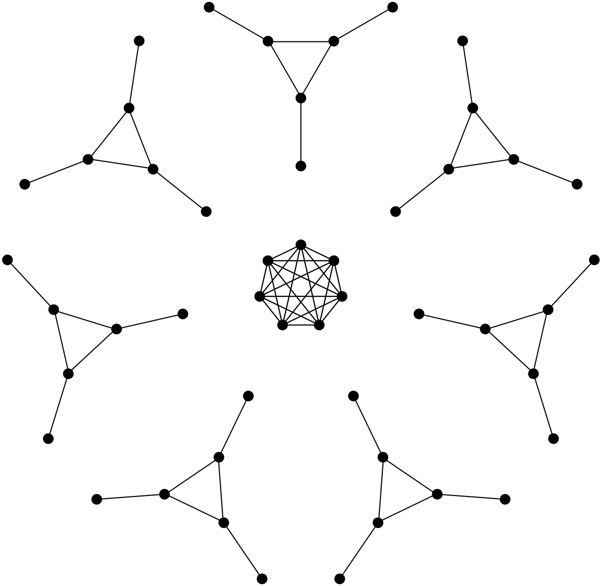

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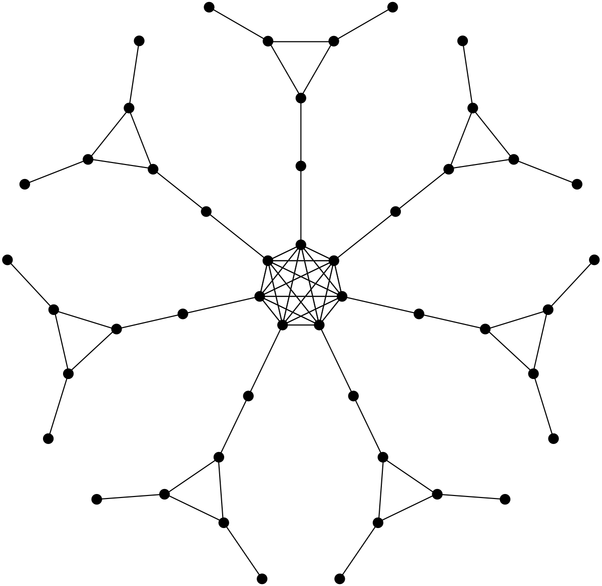

$\chi_{\mathrm{cf}}(H_k)$ and $\chi_{\mathrm{um}}(H_k)$

Prop. $\chi_{\mathrm{cf}}(H_k) = 2^{k+1} - 1$.

Prop. $2^{k+2} - 2k - 3 \leq \chi_{\mathrm{um}}(H_k) \leq 2^{k+2}$.

As a result:

$$\frac{2^{k+2} - 2k - 3}{2^{k+1} - 1} \leq \frac{\chi_{\mathrm{um}}(H_k)}{\chi_{\mathrm{cf}}(H_k)} \leq \frac{2^{k+2}}{2^{k+1} - 1}$$

and thus

$$\lim_{k \to \infty} \frac{\chi_{\mathrm{um}}(H_k)}{\chi_{\mathrm{cf}}(H_k)} = 2.$$

Bounding $\chi_{\text{um}}(G)$ by a function of $\chi_{\text{cf}}(G)$

We use the following lemma from Katchalski, McCuaig, Seager, 1995: "If the largest path subgraph of G is P_k, then $\chi_{\text{um}}(G) \leq k$." and the following from Even, Lotker, Ron, Smorodinsky, 2003: $\chi_{\text{cf}}(P_n) = 1 + \lfloor \log_2 n \rfloor$.

Prop. For every graph G, $\chi_{\text{um}}(G) \leq 2^{\chi_{\text{cf}}(G)} - 1$.

Proof.

Set $j = \chi_{\text{cf}}(G)$. Since G is j-CF-colorable, every subgraph of it is also j-CF-colorable, and in particular its longest path subgraph. This implies that the longest path subgraph in G has at most $2^j - 1$ vertices. Therefore, by the above lemma, $\chi_{\text{um}}(G) \leq 2^j - 1$. □

Computational complexity of UM and CF coloring

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CERTIFICATE CF GRAPH COLORING: Given a graph G and a coloring C of its vertices, is C a conflict-free vertex coloring of G?

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In fact, we are going to prove that Certificate CF Graph Coloring is coNP-complete.

CoNP-completeness of CF certificate

(a) CERTIFICATE CF GRAPH COLORING is in coNP because one can check that a coloring of a given graph is not conflict-free in polynomial time if given a path in which there is no vertex with a uniquely occurring color in the path.

CoNP-completeness of CF certificate

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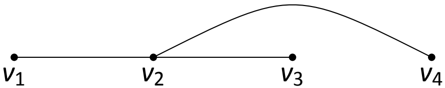
(b) We show CoNP-hardness by a reduction from the complement of the Hamiltonian path problem. For every graph G, we are going to construct in polynomial time a graph G^* and a vertex coloring C of G^* such that:

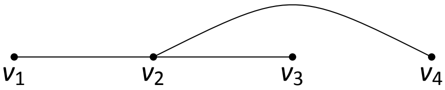
G has no Hamiltonian path
if and only if
C is a conflict-free coloring of G^*

Reduction from CoHamiltonianPath



graph G

Reduction from CoHamiltonianPath



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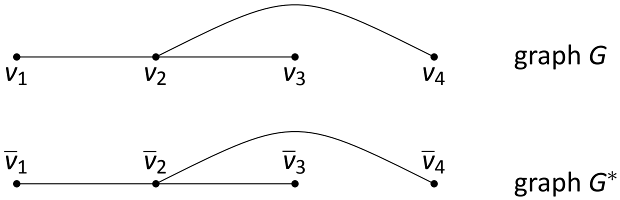

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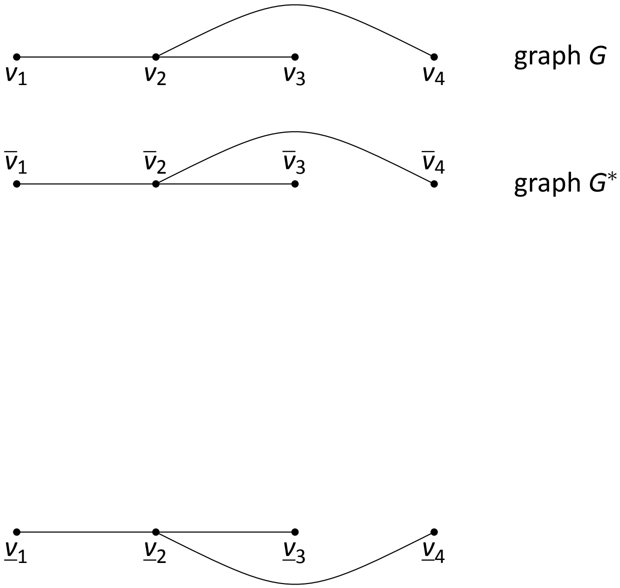

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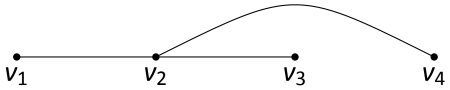
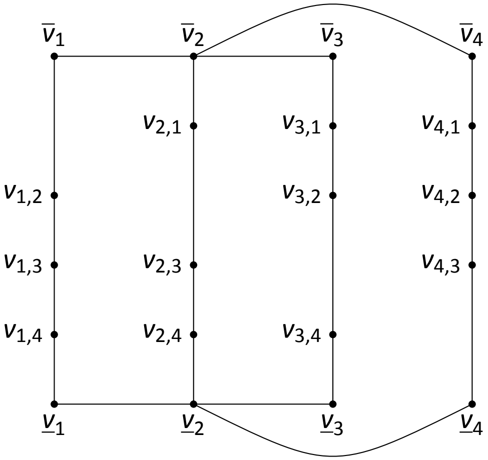
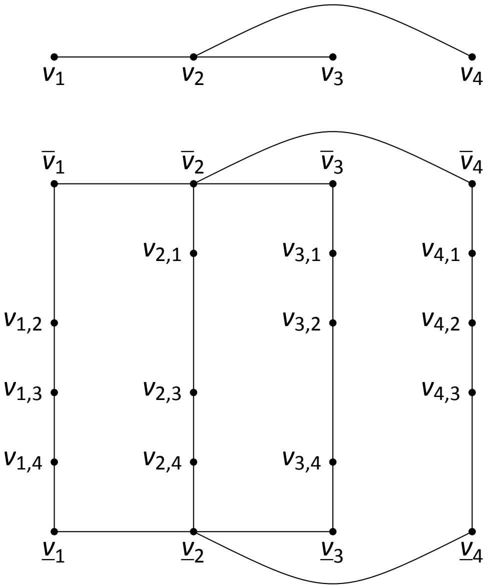


graph G



graph G^*

Reduction from CoHamiltonianPath continued



graph G

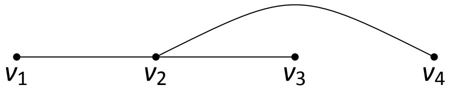
coloring C

$$C(\overline{v}_i) = C(\underline{v}_i) = i$$

for $i > j$:

$$C(v_{i,j}) = C(v_{j,i}) = n + \binom{i-1}{2} + j$$

Reduction from COHAMILTONIANPATH continued



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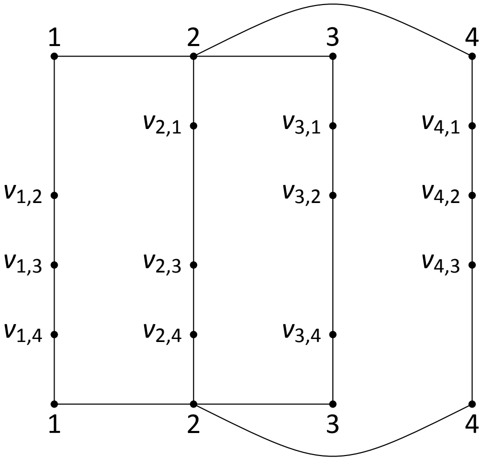

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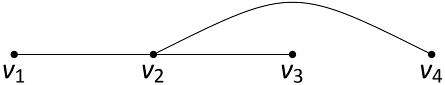

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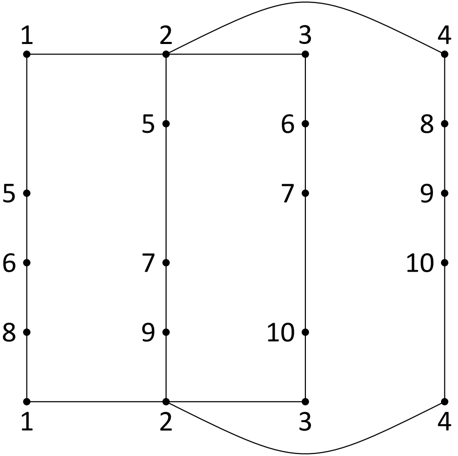

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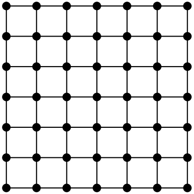
The $m \times m$ grid graph

Def. An $m \times m$ *grid* is a graph, denoted by G_m, with vertex set $\{0, \ldots, m-1\} \times \{0, \ldots, m-1\}$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| + |y_1 - y_2| \leq 1\}$

In a standard drawing of the grid graph, vertex (x, y) is drawn at point (x, y) in the plane.



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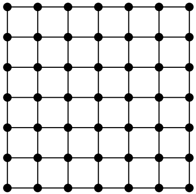
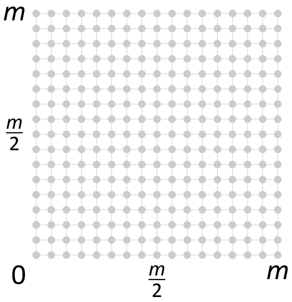


$\chi_{\mathrm{um}}(G_m) = ?, \chi_{\mathrm{cf}}(G_m) = ?$

A simple upper bound on $\chi_{\text{um}}(G_m)$



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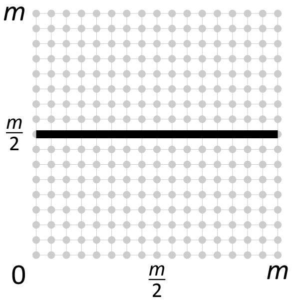

$\chi_{\text{um}}(G_m) \leq m$

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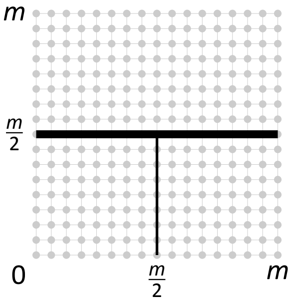

$$\chi_{\text{um}}(G_m) \leq m + m/2$$

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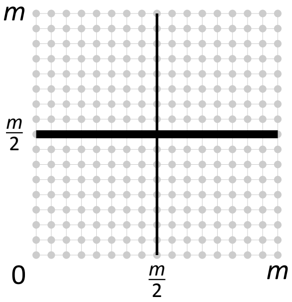

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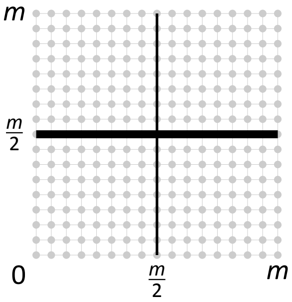

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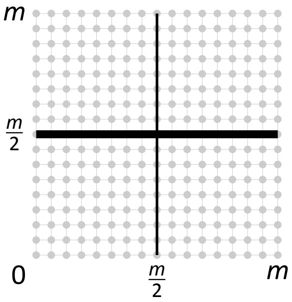

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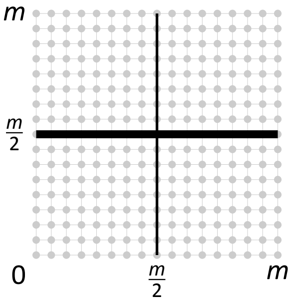

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Best known upper bound on $\chi_{\text{um}}(G_m)$ (from Bar-Noy, Ch., Lampis, Mitsou, Zachos, 2009) is roughly $2.5m$, using more intricate separators.

Method for proving a lower bound on $\chi_{\text{um}}(G_m)$

Reminder: Given a connected graph G and a UM coloring C of it, consider the set of vertices U which are colored with uniquely occurring colors in C. Then U is a separator of G.

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Thus, we can reason about a lower bound by reasoning about separators: examine cases on the size and shape of the separator formed by the highest colors of an optimal coloring and then, for each case, argue that the size of the separator plus the unique-maximum chromatic number of one of the remaining components is higher than a desired lower bound.

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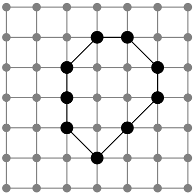

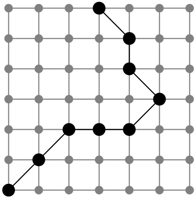

Our improved lower bound on $\chi_{\text{um}}(G_m)$

Best known lower bound before:

Prop. For $m \geq 2$, $\chi_{\text{um}}(G_m) \geq \frac{3}{2}m$.
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We improve it to:

Prop. For $m \geq 2$, $\chi_{\mathrm{um}}(G_m) \geq \frac{5}{3}m - o(m)$.

Probably, $5m/3$ is the limit of our method.

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We prove $\chi_{\text{cf}}(G_m)$ can not be a lot less than $\chi_{\text{um}}(G_m)$:

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Corollary. For $m \geq 2$, $\chi_{\text{cf}}(G_m) \geq \frac{5}{6}m - o(m)$.

Further work and open problems

properties of conflict-free coloring (monotonicity under minors maybe?)

tighten the gap between χ_{um} and χ_{cf} in general graphs

tighten the gap between lower and upper bound on $\chi_{\text{um}}(G_m)$ (1.666m lower bound, $\approx 2.5m$ upper bound)

tighten the gap between $\chi_{\text{um}}(G_m)$ and $\chi_{\text{cf}}(G_m)$

UM and CF colorings of general hypergraphs and of tree graphs w.r.t. paths (joint work with Balázs Keszegh and Dömötör Pálvölgyi)

list conflict-free coloring (joint work with Shakhar Smorodinsky)

Thank you!