Unilateral Orientation of Mixed Graphs

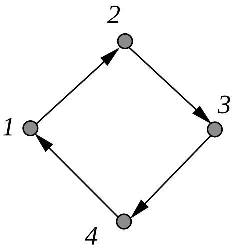
Tamara Mchedlidze, Antonios Symvonis

Dept. of Mathematics, National Technical University of Athens, Athens, Greece.
{mchet,symvonis}@math.ntua.gr

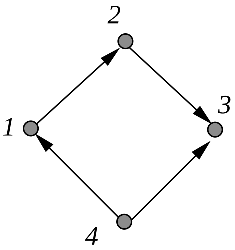
August 26, 2010

Definitions

- Strong Digraph
- Unilateral Digraph



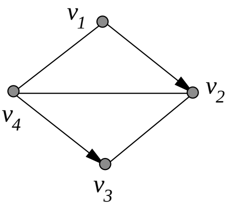
Strong digraph



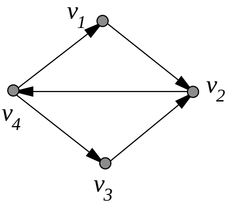
Unilateral digraph

Definitions

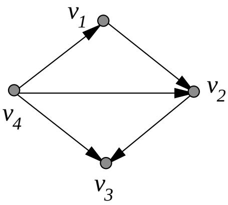
- Mixed Graph
- Strong Orientation
- Unilateral Orientation



Mixed graph G



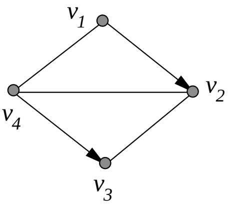
Strong orientation of G



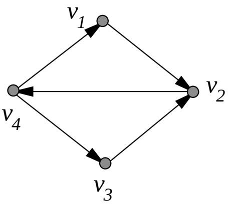
Unilateral orientation of G

Definitions

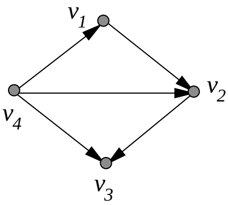
Problem: Given a mixed graph G, determine if G has a strong or a unilateral orientation.



Mixed graph G



Strong orientation of G



Unilateral orientation of G

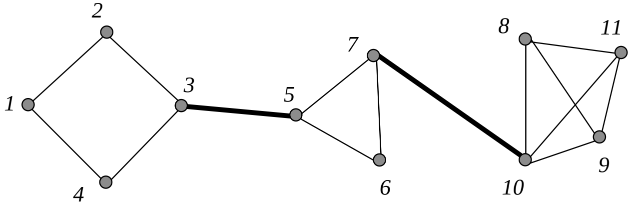
Some known results

Theorem (Robbins, 1939)

A connected graph G has a strongly connected orientation if and only if G has no bridge.

Theorem (Boesch and Tindell, 1980)

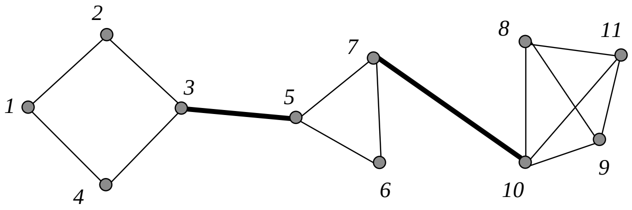
A mixed multigraph M admits a strong orientation if and only if M is strong and the underlying multigraph of M is bridgeless.



Some known results

Theorem (Chartrand, Harary, Schultz, Wall, 1994)
A connected graph G has a unilateral orientation if and only if all of the bridges of G lie on a common path.

Question: What about unilateral orientations of mixed graphs?

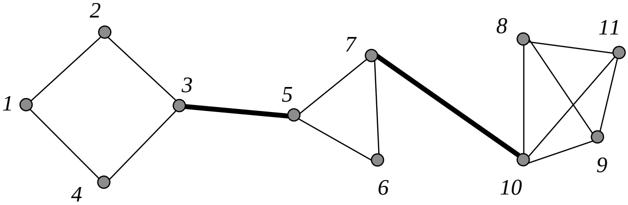


Some known results

Theorem (Chartrand, Harary, Schultz, Wall, 1994)
A connected graph G has a unilateral orientation if and only if all of the bridges of G lie on a common path.

Question: What about unilateral orientations of mixed graphs?

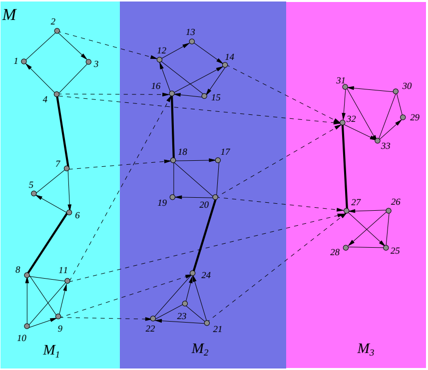
Open till now!



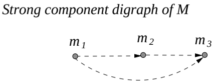
Main Results

- Characterization of Unilaterally Orientable Mixed Graphs
- Linear Time Algorithm Testing if a Given Mixed Graph has a Unilateral Orientation

Characterization of unilaterally orienable mixed graphs



(a)



(b)

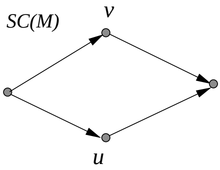
Lemma (First necessary condition)

*If a mixed graph M admits a unilateral orientation $\Rightarrow$ the **strong component digraph** of M, $SC(M)$, has a hamiltonian path.*

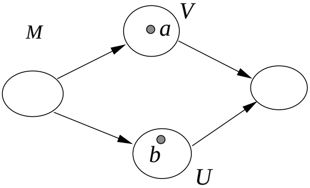
Characterization of unilaterally orientable mixed graphs

Lemma (First necessary condition)

If a mixed graph M admits a unilateral orientation $\Rightarrow$ strong component digraph of M, $SC(M)$, has a hamiltonian path.

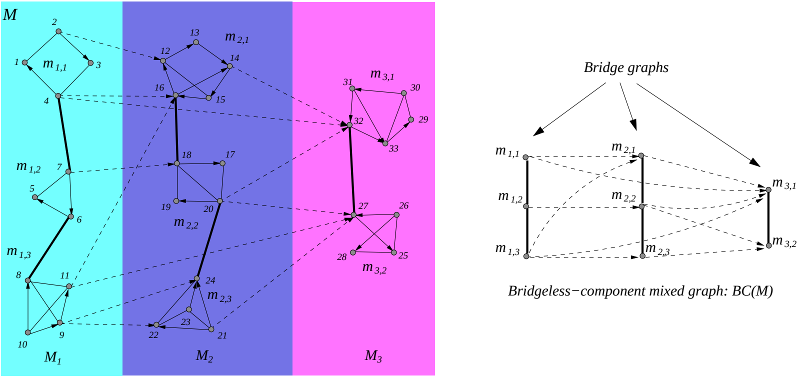


Stong component digraph without hamiltonian path



Vertices a and b are not connected by a directed path

Characterization of unilaterally orientable mixed graphs

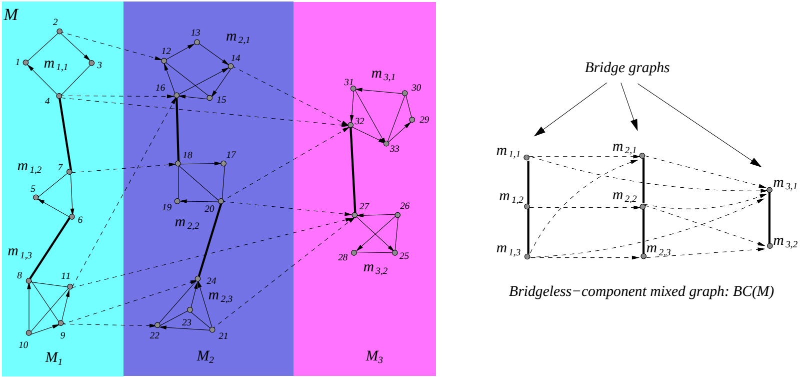


(a) (b)

Lemma (Second necessary condition)

If a mixed graph M admits a unilateral orientation ⇒
the ***bridge graph****, B(M), of each of its strong component is a path.*

Characterization of unilaterally orientable mixed graphs

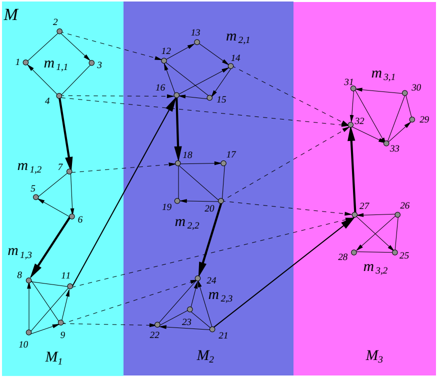


(a) (b)

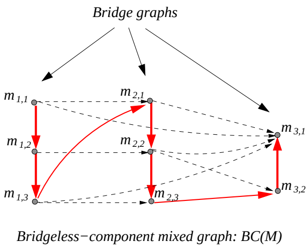
Theorem (Main result)

A mixed graph M admits a unilateral orientation ⇔ the bridgeless-component mixed graph, $BC(M)$, admits a hamiltonian orientation.

Characterization of unilaterally orienable mixed graphs



(a)

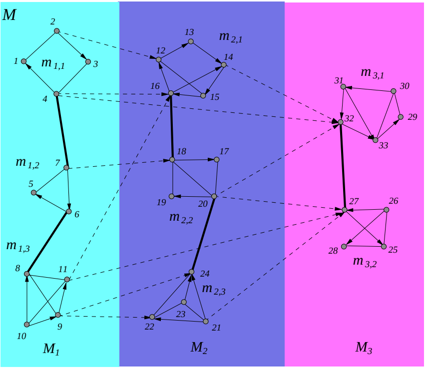


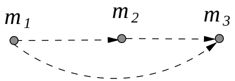
(b)

Theorem (Main result)

A mixed graph M admits a unilateral orientation $\Leftrightarrow$ the bridgeless-component mixed graph, $BC(M)$, admits a hamiltonian orientation.

The Algorithm

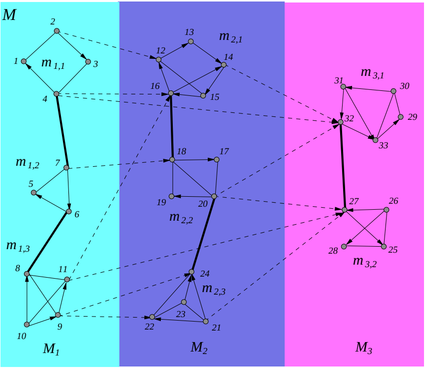


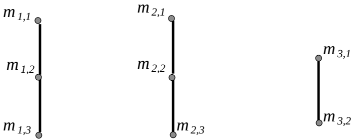


Strong component digraph SC(M)

- Construct the strong connected digraph $SC(M)$ of M;
- **if** $SC(M)$ has no hamiltonian path **then return**("NO") **else** continue;
- Complexity: $O(V + A + E)$

The Algorithm

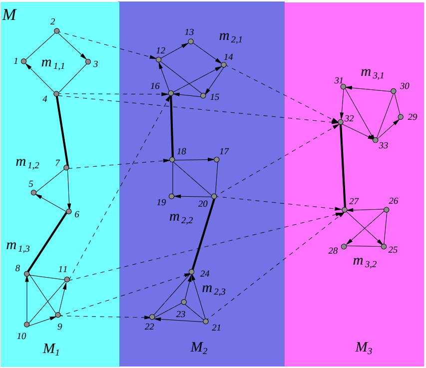


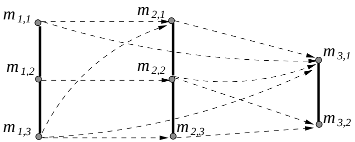


Bridge graphs of M_1 M_2 M_3

- $\forall$ strong component M_i of M, construct bridge graph $B(M_i)$;
- **if** B_{M_i} is not a simple path **then return**("NO")
  else continue;
- Complexity: $O(V + A + E)$

The Algorithm

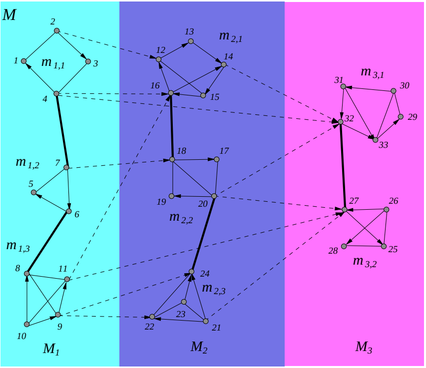


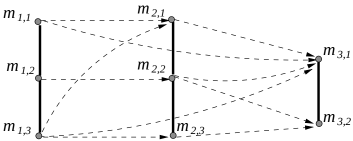


Bridgeless−component mixed graph BC(M)

- Construct bridgeless-component mixed graph $BC(M)$ of M;
- **if** $BC(M)$ has no hamiltonian path **then return**("NO") **else return**("YES");
- Complexity: $O(k)$, $k < n$.

The Algorithm





Bridgeless−component mixed graph BC(M)

- $p_1^a = p_1^b = \textbf{true}$
- $p_i^a = \left(p_{i-1}^a = \textbf{true} \wedge \exists(a_{i-1}, b_i) \in A'\right) \vee \left(p_{i-1}^b = \textbf{true} \wedge \exists(b_{i-1}, b_i) \in A'\right)$
- $p_i^b = \left(p_{i-1}^b = \textbf{true} \wedge \exists(b_{i-1}, a_i) \in A'\right) \vee \left(p_{i-1}^a = \textbf{true} \wedge \exists(a_{i-1}, a_i) \in A'\right)$, for $1 < i \leq k$

Summerizing

Theorem

Given a mixed graph $M = (V, A, E)$ *, we can decide whether* M *admits a unilateral orientation in* $O(V + A + E)$ *time. Moreover, if* M *is unilaterally orientable, a unilateral orientation can be computed in* $O(V + A + E)$ *time.*

Theorem (Our result)

A mixed graph admits a unilateral orientation if and only if all the bridges of its strong components lie on a common path.

Theorem (Chartrand, Harary, Schultz, Wall, 1994)

A connected graph G *has a unilateral orientation if and only if all of the bridges of* G *lie on a common path.*

Thank You!