Coordination Mechanisms for Weighted Sum of Completion Times in Machine Scheduling

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1. Machine Scheduling
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Machine Scheduling

- We have a set N of n jobs and a set M of m machines
- Each job needs to be assigned to exactly one machine
- Each machine can process only one job at any time
- A schedule defines which job will be processed by each machine at any point
- For each job i, we use the following notation:
  - It's processing time on machine j is denoted by p_{ij}
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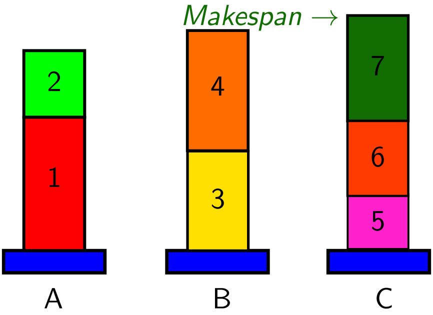
Objective Functions

- Makespan ($\max_i c_i$)
- Sum of completion times ($\sum_i c_i$)
- Weighted sum of completion times ($\sum_i w_i c_i$)



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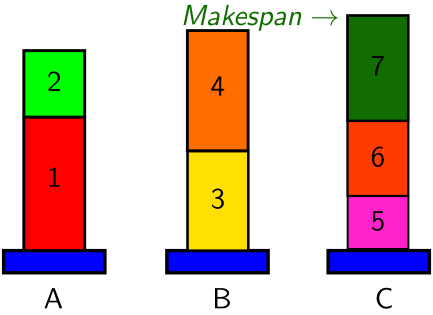

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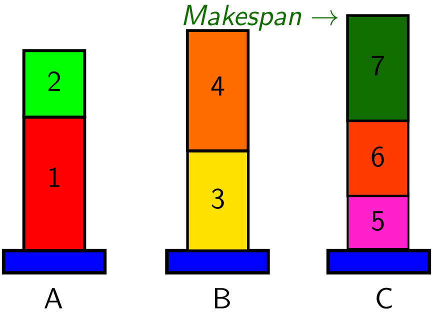

Machine Models

- Identical machines
  - Each job i has a processing requirement p_i
  - The processing time of job i on any machine j will be $p_{ij} = p_i$
- Related machines
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  - The processing time of job i on machine j will then be $p_{ij} = \frac{p_i}{q_j}$
- Restricted machines
  - Each job i has a processing requirement p_i
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- Minimizing $\sum_i c_i$ is in P even for unrelated machines [H 73, BCS 74]
- Minimizing $\sum_i w_i c_i$ is NP-hard even for identical machines [LKB 77]
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- Let each job be controlled by a selfish agent
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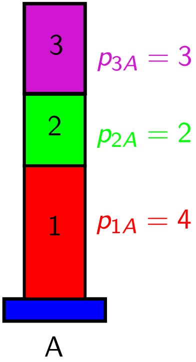
Strongly Local Policies

For example ShortestFirst and EqualSharing:



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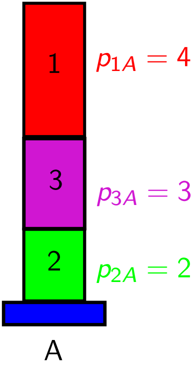
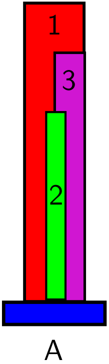
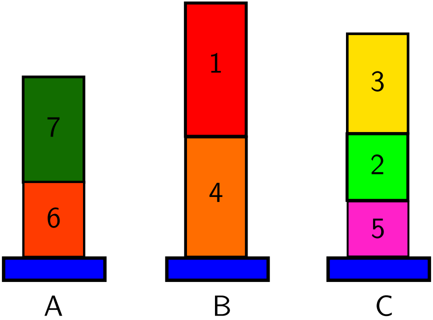




Coordination Mechanism

- By Christodoulou, Koutsoupias and Nanavati [ICALP 04]
- A set of local policies, one for each machine



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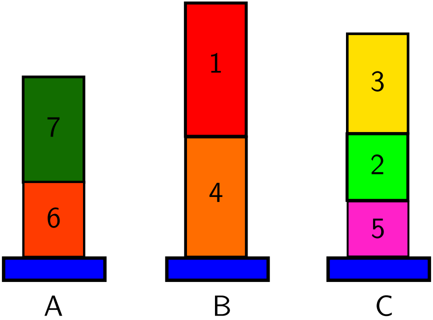

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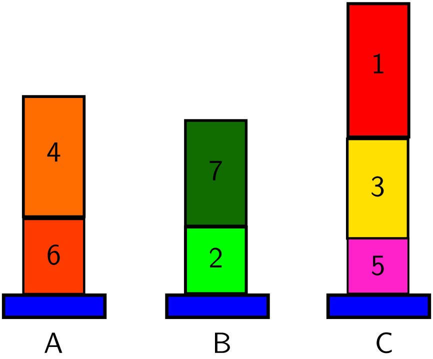

Normal Form Game

- Assume that all the job weights are equal to 1
- Given a coordination mechanism α we have defined a game
- Each assignment (strategy profile) s implies a completion time or cost denoted by $c_i^\alpha(s)$ for each player i
- An assignment s is a Pure Nash Equilibrium (PNE) if:

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Price of Anarchy

- Defined by Koutsoupias and Papadimitriou [STACS 99]
- The PoA of the induced game w.r.t. the sum of completion times is:

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- Then [ILMS 05]
  - Study and survey results for four coordination mechanisms and four machine models
  - Study convergence time and existence of PNE
- Next [AJM 08]:
  - Prove that strongly local ordering policies are $\Omega(m)$
  - Present a local policy that achieves $O(\log m)$ but doesn't induce potential games
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  - Presents three new coordination mechanisms for unrelated machines
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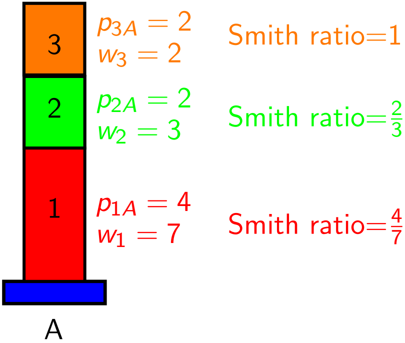
- Non-preemptive policy
- Each machine j gives higher priority to jobs with smaller $\frac{p_{ij}}{w_i}$
- In the optimal solution, every machine must follow this policy



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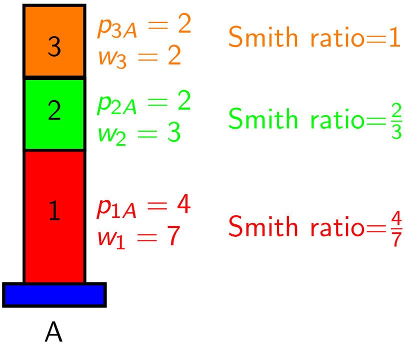

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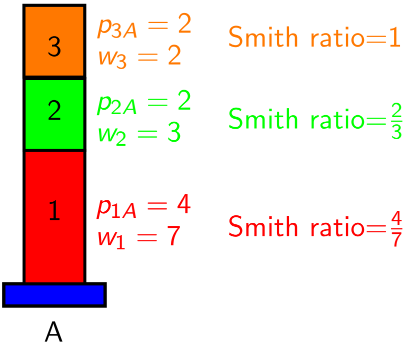

PoA for weighted sum of completion times

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Robust PoA

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A coordination mechanism α is defined to be (λ, μ)-smooth if for every two assignments s and s^* of any game that it may induce

$$\sum_{i \in N} w_i c_i^{\alpha}(s_{-i}, s_i^*) \leq \lambda \sum_{i \in N} w_i c_i^{\mathsf{SR}}(s^*) + \mu \sum_{i \in N} w_i c_i^{\alpha}(s).$$

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The *Robust PoA* of a coordination mechanism is equal to $\inf\left\{\frac{\lambda}{1-\mu} : (\lambda, \mu) \text{ s.t. the mechanism is } (\lambda, \mu)\text{-smooth}\right\}$.

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The *Robust PoA* of a coordination mechanism is equal to $\inf \left\{ \frac{\lambda}{1-\mu} : (\lambda, \mu) \text{ s.t. the mechanism is } (\lambda, \mu)\text{-smooth} \right\}$.

Theorem

The Robust PoA of SmithRule for unrelated machines is at most 4.

We show that this coordination mechanism is $(2, \frac{1}{2})$-smooth by showing that for any two assignments s and s^*:

$$\sum_{i \in N} w_i c_i^{SR}(s_{-i}, s_i^*) \leq 2C^{SR}(s^*) + \frac{1}{2}C^{SR}(s).$$

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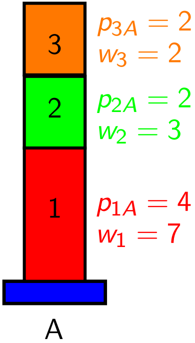
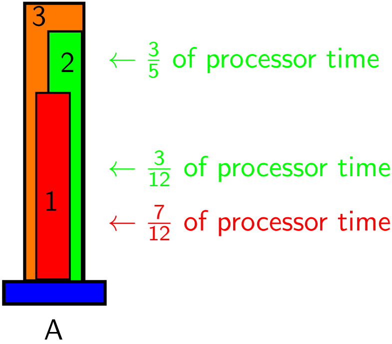
Theorem

The pure PoA of *any set of strongly local ordering policies* for restriced identical machines is at least 4. This is true even for the unweighted case. (Generalizing [CFKKM 06] and [CQ 10])

Proportional Sharing

- A generalization of the EqualSharing policy for weighted jobs
- Each job gets a share of the processing time equal to the ratio of its weight over the sum of the weights of all jobs being processed on the same machine at that time





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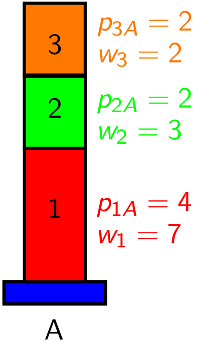

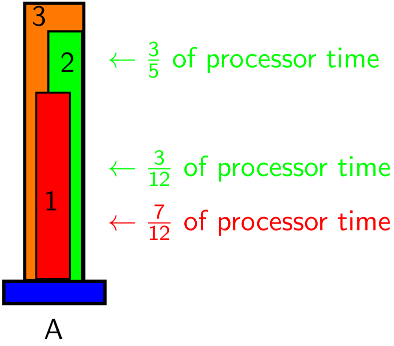

EqualSharing

Theorem

The robust PoA of EqualSharing for unrelated machines is at most 2.5.
This bound is tight even for the restricted related machines model [CFKKM 06].

We show that this coordination mechanism is $(5/3, 1/3)$-smooth by showing that for any two assignments s and s^*:

$$\sum_{i \in N} c_i^{ES}(s_{-i}, s_i^*) \leq \frac{5}{3} \sum_{i \in N} c_i^{SF}(s^*) + \frac{1}{3} \sum_{i \in N} c_i^{ES}(s).$$

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Theorem

The ProportionalSharing coordination mechanism induces exact potential games.

We show that $\Phi(s) = \frac{1}{2}\sum_{i' \in N} w_{i'} \left(c_{i'}(s) + p_{i' s_{i'}} \right)$ serves as an exact potential function for these games.

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Theorem

The robust PoA of ProportionalSharing for unrelated machines is at most $\phi + 1 = \frac{3+\sqrt{5}}{2} \approx 2.618$.
This bound is tight even for the restricted related machines model [CFKKM 06].

We show that this coordination mechanism is $\left(\frac{\phi+2}{2}, \frac{1}{2\phi}\right)$-smooth by showing that for any two assignments s and s^*:

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Approximation Algorithm

- For unrelated machines and weighted sum of completion times, the minimization problem is NP-hard [LKB 77]
- First constant factor ($\frac{16}{3}$) approximation algorithm [HSSW 97]
- Later improved to $\frac{3}{2} + \epsilon$ [SS 02]
- Independently further improved to $\frac{3}{2}$ by [SS 99, S 01]

The only known constant factor approximation algorithms are based on LP or convex quadratic program relaxations!

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Approximation Algorithm

- We know that a PNE always exists for ProportionalSharing
- For any such PNE s we know that $\sum_i w_i c_i^{PS}(s) \leq 2.619$ OPT
- Computing such a PNE implies a 2.619-approx. algorithm
- In general, best response dynamics might need an exponential number of deviations before we arrive at a PNE

A coordination mechanism with potential function Φ and social cost function C is said to be β-nice if for any configuration s:

- $\Phi(s) \leq C(s)$
- $C(s) \leq \beta \mathrm{OPT} + 2\sum_i (c_i(s) - c_i(s_{-i}, s_i'))$

Lemma

The ProportionalSharing coordination mechanism is β-nice with $\beta = \frac{3+\sqrt{5}}{2} \approx 2.618$.

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Corollary

Starting from some initial configuration s^0 and moving the player with the maximum absolute improvement in each step, leads to a profile s with $C^{PS}(s) \leq (2.619 + O(\epsilon))C^{SR}(s^*)$ in at most $O\left(\frac{n}{\epsilon} \log\left(\frac{\Phi(s^0)}{\Phi(s^*)}\right)\right)$ steps.

Conclusion

- SmithRule: Robust PoA is at most 4
- For any set of strongly local ordering policies the pure PoA is at least 4
- EqualSharing: Robust PoA is 2.5
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