Infinite Automata, Logics and Games

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ω-Automata

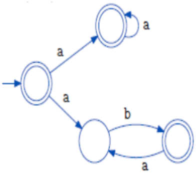
Nondeterministic Tree Automata

Ehrenfeucht-Fraïssé Games

A nondeterministic finite automaton (*NFA*) is a 5-tuple, $(Q, \Sigma, \Delta, q_0, F)$, consisting of

- a finite set of states Q,
- a finite set of input symbols Σ,
- a transition function $\Delta : Q \times \Sigma \to P(Q)$,
- an initial state $q_0 \in Q$,
- a set of states F distinguished as accepting (or final) states $F \subseteq Q$.

NFA for $a^* + (ab)^*$:



REG is the class of languages recognised by a finite automaton

An ω-automaton is a quintuple $(Q, \Sigma, \delta, q_I, Acc)$, where

- Q is a finite set of states,
- Σ is a finite alphabet,
- $\delta : Q \times \Sigma \rightarrow P(Q)$ is the state transition function,
- $q_I \in Q$ is the initial state,
- Acc is the acceptance component.

In a deterministic ω-automaton, a transition function $\delta : Q \times \Sigma \rightarrow Q$ is used.

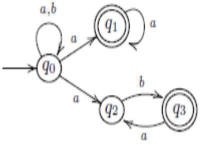
Let $A = (Q, \Sigma, \delta, q_I, Acc)$ be an ω-automaton. A run of A on an ω-word $\alpha = a_1 a_2 ... \in \Sigma^\omega$ is an infinite state sequence $\rho = \rho(0)\rho(1)\rho(2)... \in Q^\omega$, such that the following conditions hold:

1. $\rho(0) = q_I$
2. $\rho(i) \in \delta(\rho(i-1), a_i)$ for $i \geqslant 1$ if A is nondeterministic,
   $\rho(i) = \delta(\rho(i-1), a_i)$ for $i \geqslant 1$ if A is deterministic.

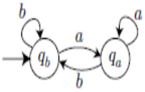
For a run ρ of an ω-automaton, let $Inf(\rho) = \{q \in Q : \forall i \exists j > i \, \rho(j) = q\}$.

An ω-automaton $A = (Q, \Sigma, \delta, q_I, Acc)$ is called

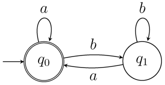
- **Büchi** automaton if $Acc = F \subseteq Q$ and the acceptance condition is the following: A word $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α satisfying the condition: $Inf(\rho) \cap F \neq \emptyset$.
- **Muller** automaton if $Acc = \mathcal{F} \subseteq P(Q)$ and the acceptance condition is the following: A word $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α satisfying the condition: $Inf(\rho) \in F$.
- **Rabin** automaton if $Acc = \{(E_1, F_1), ..., (E_k, F_k)\}$, with $E_i, F_i \subseteq Q$, $1 \leqslant i \leqslant k$, and the acceptance condition is the following: A word $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α satisfying the condition: $\exists (E, F) \in Acc(Inf(\rho) \cap E = \emptyset) \wedge (Inf(\rho) \cap F \neq \emptyset)$.
- **Streett** automaton if $Acc = \{(E_1, F_1), ..., (E_k, F_k)\}$, with $E_i, F_i \subseteq Q$, $1 \leqslant i \leqslant k$, and the acceptance condition is the following: A word $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α satisfying the condition: $\forall (E, F) \in Acc(Inf(\rho) \cap E \neq \emptyset) \vee (Inf(\rho) \cap F = \emptyset)$ $\big($or $\forall (E, F) \in Acc(Inf(\rho) \cap F \neq \emptyset) \rightarrow (Inf(\rho) \cap E \neq \emptyset)\big)$.



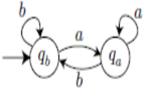
Büchi automaton for $(a + b)^* a^\omega + (a + b)^* (ab)^\omega$ with $F = \{q_1, q_3\}$



Muller automaton for $(a + b)^* a^\omega + (a + b)^* b^\omega$ with $\mathcal{F} = \{\{q_a\}, \{q_b\}\}$



Rabin automaton for $(a + b)^* a^\omega$ with $Acc = \{(\{q_1\}, \{q_0\})\}$



Streett automaton with $Acc = \{(\{q_b\}, \{q_a\})\}$.
Each word in the accepted language contains infinitely many a's only if it contains infinitely many b's (or equivalently they have finitely many a's or infinitely many b's).

The Büchi recognizable ω-languages are the ω-languages of the form

$$L = \bigcup_{i=1}^{k} U_i V_i^{\omega} \text{ with } k \in \omega \text{ and } U_i, V_i \in REG \text{ for } i = 1, \dots, k.$$

This family of ω-languages is also called the **ω-Kleene closure** of the class of regular languages and are commonly referred to as ω-REG.

The **emptiness problem** for Büchi automata is decidable.

Muller automata are equally expressive as nondeterministic Büchi automata.

Proof: On the board.

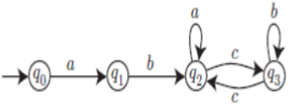
Rabin automata and Streett automata are equally expressive as Muller automata.

Proof:

- For a Rabin automaton $A = (Q, \Sigma, \delta, q_I, Acc)$, define the Muller automaton $A' = (Q, \Sigma, \delta, q_I, \mathcal{F})$, where
  $\mathcal{F} = \{G \in P(Q) | \exists (E, F) \in Acc. \, G \cap E = \emptyset \wedge G \cap F \neq \emptyset\}$.
  For a Streett automaton $A = (Q, \Sigma, \delta, q_I, Acc)$, define the Muller automaton $A' = (Q, \Sigma, \delta, q_I, \mathcal{F})$, where
  $\mathcal{F} = \{G \in P(Q) | \forall (E, F) \in Acc. \, G \cap E \neq \emptyset \vee G \cap F = \emptyset\}$.
- Conversely, given a Muller automaton, transform it into a nondeterministic Büchi automaton.
  Büchi acceptance can be viewed as a special case of Rabin acceptance, where $Acc = \{(\emptyset, F)\}$, as well as a special case of Streett acceptance, where $Acc = \{(F, Q)\}$.

An ω-automaton $A = (Q, \Sigma, \delta, q_I, c)$ with acceptance component $c : Q \to \{1, ..., k\}$ (where $k \in \omega$) is called **parity** automaton if it is used with the following acceptance condition:
An ω-word $\alpha \in \Sigma^\omega$ is accepted by A iff there exists a run ρ of A on α with

$$min\{c(q)|q \in Inf(\rho) \text{ is even}\}$$



Parity automaton A with colouring function c defined by $c(q_i) = i$.
$L(A) = ab(a^*cb^*c)^*a^\omega$

.

Parity automata can be converted into Rabin automata.

Proof: Let $A = (Q, \Sigma, \delta, q_I, c)$ be a parity automaton with $c : Q \to \{0, ..., k\}$. An equivalent Rabin automaton $A' = (Q, \Sigma, \delta, q_I, Acc)$ has the acceptance component $Acc = \{(E_0, F_0), ..., (E_r, F_r)\}, r = \lfloor \frac{k}{2} \rfloor$,
$E_i = \{q \in Q | c(q) < 2i\}$ and $F_i = \{q \in Q | c(q) \leqslant 2i\}$.

Muller automata can be converted into parity automata (a special case of Rabin automata).

Proof: On the board.

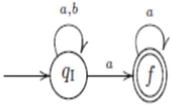
- Nondeterministic Büchi, Muller, Rabin, Streett, and parity automata are all equivalent in expressive power, i.e. they recognize the same ω-languages.
- The ω-languages recognized by these ω-automata form the class ω-KC(REG), i.e. the ω-Kleene closure of the class of regular languages.

- NFAs are equivalent to DFAs.
- NPDAs are not equivalent to DPDAs.
- Nondeterministic ω-automata are equivalent to deterministic ones?

Deterministic vs Nondeterministic Büchi Automata

There exist languages which are accepted by some nondeterministic Büchi-automaton but not by any deterministic Büchi automaton.

Proof. The following automaton is a nondeterministic Büchi automaton for $L = (a+b)^* a^\omega$.



Assume that there is a deterministic Büchi automaton A for the language L. Then there exist $n_0, n_1, n_2, \ldots$ such that A accepts the ω-word $w = a^{n_0} b a^{n_1} b a^{n_2} b \ldots \notin L$.

- Deterministic Muller, Rabin, Streett, and parity automata recognize the same ω-languages.
- The class of ω-languages recognized by any of these types of ω-automata is closed under complementation.

Proof:

- The transformations between nondeterministic automata work for deterministic ones except for those that use nondeterministic Büchi automata.

  NRabin $\longrightarrow$ NStreett: NRabin $\longrightarrow$ NMuller $\longrightarrow$ NBüchi $\longrightarrow$ NStreett

  DRabin $\longrightarrow$ DStreett: DRabin for L $\longrightarrow$ DMuller for L $\longrightarrow$ DMuller for $\overline{L}$ $\longrightarrow$ DRabin for $\overline{L}$ $\longrightarrow$ DStreett for L
- The languages recognizable by deterministic Muller automata are closed under union, intersection and complementation.

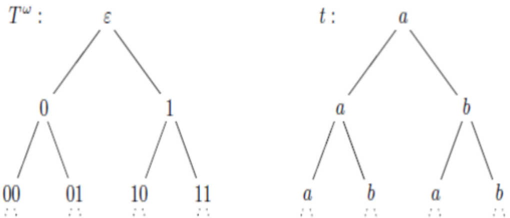
Determinization of Büchi Automata

Every nondeterministic Büchi automaton can be transformed into an equivalent deterministic Muller automaton (or a deterministic Rabin automaton).

- The powerset construction fails in case of Büchi automata.
- In 1963 Muller presented a faulty construction.
- In 1966 McNaughton showed that a Büchi automaton can be transformed effectively into an equivalent deterministic Muller automaton.
- Safra's construction of 1988 leads to deterministic Rabin or Muller automata: given a nondeterministic Büchi automaton with n states, the equivalent deterministic automaton has $2^{\mathcal{O}(nlogn)}$ states.
- For Rabin automata, Safra's construction is optimal. The question whether it can be improved for Muller automata is open.
- In 1995 Muller and Schupp presented a 'more intuitive' alternative, which is also optimal for Rabin automata.

All the above ω-automata, except for deterministic Büchi, recognize the same ω-languages.

- The **infinite binary tree** is the set $T^\omega = \{0,1\}^*$ of all finite words on $\{0,1\}$.
- The elements $u \in T^\omega$ are the **nodes** of T^ω where ϵ is the root and $u0, u1$ are the immediate left and right successors of node u.
- An ω-word $\pi \in \{0,1\}^\omega$ is called a **path** of the binary tree T^ω.
- The set of all Σ**-labelled** trees, T^ω_Σ, contains trees where each node is labelled with a symbol of the alphabet Σ, i.e. trees with a mapping $t : T^\omega \to \Sigma$.



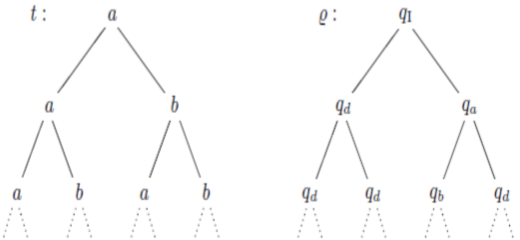
A **Muller tree automaton** is a quintuple $A = (Q, \Sigma, \Delta, q_I, \mathcal{F})$, where

- Q is a finite state set,
- Σ is a finite alphabet,
- $\Delta \subseteq Q \times \Sigma \times Q \times Q$ denotes the transition relation,
- q_I is an initial state,
- $\mathcal{F} \subseteq P(Q)$ is a set of designated state sets.

- A **run** of A on an input tree $t \in T_\Sigma$ is a tree $\rho \in T_Q$, satisfying $\rho(\epsilon) = q_I$ and for all $w \in \{0,1\}^*$: $(\rho(w), t(w), \rho(w0), \rho(w1)) \in \Delta$.
- A run is called **successful** if for each path $\pi \in \{0,1\}^\omega$ the Muller acceptance condition is satisfied, that is, if $Inf(\rho|\pi) \in \mathcal{F}$.
- A accepts the tree t if there is a successful run of A on t.
- The tree language recognized by A is the set $T(A) = \{t \in T^\omega | A \text{ accepts } t\}$.

$A = (\{q_I, q_a, q_b, q_d\}, \{a, b\}, \Delta, q_I, \mathcal{F})$, where Δ includes:

$$(q_I, a, q_a, q_d), (q_I, a, q_d, q_a), (q_I, b, q_b, q_d), (q_I, b, q_d, q_b),$$
$$(q_d, a, q_d, q_d), (q_d, b, q_d, q_d),$$
$$(q_a, b, q_b, q_d), (q_a, b, q_d, q_b), (q_a, a, q_I, q_d), (q_a, a, q_d, q_I),$$
$$(q_b, a, q_a, q_d), (q_b, a, q_d, q_a), (q_b, b, q_I, q_d), (q_b, b, q_d, q_I).$$



First transitions of ρ

The Muller tree automaton $A = (\{q_I, q_a, q_b, q_d\}, \{a, b\}, \Delta, q_I, \mathcal{F})$, where Δ includes:

$$\begin{gathered}(q_I, a, q_a, q_d), (q_I, a, q_d, q_a), (q_I, b, q_b, q_d), (q_I, b, q_d, q_b),\\ (q_d, a, q_d, q_d), (q_d, b, q_d, q_d),\\ (q_a, b, q_b, q_d), (q_a, b, q_d, q_b), (q_a, a, q_I, q_d), (q_a, a, q_d, q_I),\\ (q_b, a, q_a, q_d), (q_b, a, q_d, q_a), (q_b, b, q_I, q_d), (q_b, b, q_d, q_I).\end{gathered}$$

and $\mathcal{F} = \{\{q_a, q_b\}, \{q_d\}\}$ recognizes the tree language
$T = \{t \in T_{\{a,b\}} |$ there is a path π through t such that $t|\pi \in (a+b)^*(ab)^\omega\}$.

The Muller tree automaton $A = (\{q_I, q_1, q_2\}, \{a, b\}, \Delta, q_I, \{\{q_I\}\})$, where Δ includes the transitions:

$$(q_I, a, q_I, q_I),\ (q_I, b, q_1, q_1),$$
$$(q_1, b, q_1, q_1), (q_1, a, q_I, q_I).$$

recognizes the tree language
$T = \{t \in T_{\{a,b\}} \mid$ any path through t carries only finitely many $b's\}$.

The above language T can not be recognized by a Büchi tree automaton.

Büchi tree automata are strictly weaker than Muller tree automata.

Muller, Rabin, Streett, and parity tree automata all recognize the same tree languages.

- **Games on Sets**
  Let A, B be sets, i.e. $\sigma = \emptyset$. Let also $\|A\|, \|B\| \geqslant n$.
  Then $A \equiv_n B$.

Proof. Suppose after i rounds that the position is $((a_1, ..., a_i), (b_1, ..., b_i))$.
When the spoiler picks an element $a_{i+1} \in |A|$, then if

1. $a_{i+1} = a_j$ for $j \leqslant i$, then the duplicator responds with $b_{i+1} = b_j$.
2. otherwise, the duplicator responds with any $b_{j+1} \in |B| - \{b_1, ..., b_i\}$, which exists since $\|B\| \geqslant n$.

- **Games on Linear Orders**
  Let $k > 0$, and let L_1, L_2 be linear orders of length at least 2^k.
  Then $L_1 \equiv_k L_2$.

Proof. Let $|L_1| = \{1, ..., n\}$ and $|L_2| = \{1, ..., m\}$, with $n, m \geqslant 2^k + 1$, and $\sigma' = \{<, min, max\}$.
Let $\mathbf{a} = (a_{-1}, a_0, a_1, ..., a_i)$ and $\mathbf{b} = (b_{-1}, b_0, b_1, ..., b_i)$ after round i. Then, the duplicator can play in such a way that the following hold for $-1 \leqslant j, l \leqslant i$ after each round i:

1. 1 if $d(a_j, a_l) < 2^{k-i}$, then $d(b_j, b_l) = d(a_j, a_l)$.
2. if $d(a_j, a_l) \geqslant 2^{k-i}$, then $d(b_j, b_l) \geqslant 2^{k-i}$.
3. $a_j \leqslant a_l \Leftrightarrow b_j \leqslant b_l$.

Proof continued. The base case of $i = 0$ is immediate.
For the induction step, suppose the spoiler is making his $(i+1)$st move in L_1, such that $a_j < a_{i+1} < a_l$. By condition 3 of the inductive hypothesis $b_j < b_{i+1} < b_l$.
There are two cases:

- $d(a_j, a_l) < 2^{k-i}$. By the inductive hypothesis $d(b_j, b_l) = d(a_j, a_l)$. The duplicator finds b_{i+1} so that $d(a_j, a_{i+1}) = d(b_j, b_{i+1})$ and $d(a_{i+1}, a_l) = d(b_{i+1}, b_l)$.
- $d(a_j, a_l) \geqslant 2^{k-i}$. By inductive hypothesis $d(b_j, b_l) \geqslant 2^{k-i}$.
  We have three possibilities:
  1. $d(a_j, a_{i+1}) < 2^{k-(i+1)}$. Then $d(a_{i+1}, a_l) \geqslant 2^{k-(i+1)}$, and the duplicator chooses b_{i+1} so that $d(b_j, b_{i+1}) = d(a_j, a_{i+1})$ and $d(b_{i+1}, b_l) \geqslant 2^{k-(i+1)}$.
  2. $d(a_{i+1}, a_l) < 2^{k-(i+1)}$. This case is similar to the previous one.
  3. $d(a_j, a_{i+1}) \geqslant 2^{k-(i+1)}$, $d(a_{i+1}, a_l) \geqslant 2^{k-(i+1)}$). Since $d(b_j, b_l) \geqslant 2^{k-i}$, by choosing b_{i+1} to be the middle of the interval $[b_j, b_l]$, duplicator ensures that $d(b_j, b_{i+1}) \geqslant 2^{k-(i+1)}$ and $d(b_{i+1}, b_l) \geqslant 2^{k-(i+1)}$.

Ehrenfeucht-Fraïssé Theorem

Let A and B be two σ-structures, where σ is a relational vocabulary. Then the following are equivalent:

1. A and B agree on $FO[k]$.
2. $A \equiv_k B$.

Corollary

A property P of finite σ-structures is not expressible in FO if for every $k \in \mathbb{N}$, there exist two finite σ-structures, A_k and B_k, such that:

- $A_k \equiv_k B_k$, and
- A_k has property P, and B_k does not.

EVEN is not FO-expressible over linear orders.

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Corollary2

A property P is expressible in FO iff there exists a number k such that for every two structures A, B, if $A \in P$ and $A \equiv_k B$, then $B \in P$.

Proof.

- If P is expressible by an FO sentence Φ, let $k = qr(\Phi)$. If $A \in P$, then $A \models \Phi$, and hence for B with $A \equiv_k B$, we have $B \models \Phi$. Thus, $B \in P$.
- If $A \in P$ and we force A to agree on all $FO[k]$ sentences with B, then $B \in P$. A and B have the same rank-k type, and hence P is a union of types, and thus definable by a disjunction of some of the α_K's.

Ehrenfeucht-Fraïssé Theorem

Let A and B be two σ-structures, where σ is a relational vocabulary. Then the following are equivalent:

1. A and B agree on $FO[k]$.
2. $A \simeq_k B$.

Proof. $1 \Rightarrow 2$: Assume A and B agree on all quantifier-rank $k+1$ sentences. For the *forth* condition: Pick $a \in |A|$, and let α_i be its rank-k 1-type. Then $A \models \exists x \alpha_i(x)$, where $\exists x \alpha_i(x)$ is a sentence of quantifier-rank $k+1$. Hence $B \models \exists x \alpha_i(x)$. Let b be the witness for the existential quantifier, that is, $tp_k(A, a) = tp_k(B, b)$. Equivalently for every ψ with $qr(\psi) = k$, $A \models \psi$ iff $B \models \psi$. By inductive hypothesis, $(A, a) \simeq_k (B, b)$.

$2 \Rightarrow 1$: Assume $A \simeq_{k+1} B$. Every $FO[k+1]$ sentence is a boolean combination of $\exists x \phi(x)$, where $\phi \in FO[k]$.

Assume that $A \models \exists x \phi(x)$, so $A \models \phi(a)$ for some $a \in |A|$. By forth, find $b \in |B|$ such that $(A, a) \simeq_k (B, b)$. By inductive hypothesis, (A, a) and (B, b) agree on $FO[k]$. Hence, $B \models \phi(b)$, and thus $B \models \exists x \phi(x)$.