Turing Machines and The Chomsky Hierarchy

November 24, 2011
Grammars

Grammar $G = (\Sigma, N, S, R)$
- $\Sigma$: set of terminal symbols
- $N$: set of non-terminal symbols
- $S \in N$: start symbol
- $R \subseteq (\Sigma \cup N)^* \times (\Sigma \cup N)^*$: finite set of rules
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- Relation $\rightarrow \subseteq (\Sigma \cup N)^* \times (\Sigma \cup N)^*$
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- Relation $\rightarrow \subseteq (\Sigma \cup N)^* \times (\Sigma \cup N)^*$
- Relation $\rightarrow^* \subseteq (\Sigma \cup N)^* \times (\Sigma \cup N)^*$ is the reflexive-transitive closure of $\rightarrow$
Languages generated by grammars

Lemma

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We can enumerate all possible derivations of strings from $S$

1. $L := [S]$
2. Pop out the first element of $L$ (call it $x$)
3. if $x \in \Sigma^*$ then print $x$
   else for each derivation rule applicable on $x$ add the result of the application of the rule as the last element of $L$
4. if $L \neq [ ]$ go to step 2
Languages generated by grammars

Grammar derivations can be used to simulate the moves of a Turing Machine, where the string being manipulated represents the Turing Machine’s configuration.

- we define non-terminal symbols $R, L$
- For all $((q, \sigma), (q', \sigma', \rightarrow)) \in \delta$ we create the derivations $S\sigma, q, \sigma_i \rightarrow R\sigma'\sigma_i, q$, for all $\sigma_i$
- For all $((q, \sigma), (q', \sigma', \leftarrow)) \in \delta$ we create the derivation $S\sigma, q, \rightarrow L, q', \sigma'$
- For all $((q, \sigma), (q', \sigma', -)) \in \delta$ we create the derivation $S\sigma, q \rightarrow S\sigma', q'$
- We create the derivations $\sigma_i L \rightarrow S\sigma_i$ for all $\sigma_i$
- We create the derivations $R\sigma_i \rightarrow \sigma_i S$ for all $\sigma_i$
- For all $((q, \sigma), (yes, \sigma', m))$ we create the derivation $S\sigma, q, \rightarrow \sigma'$
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Given grammar $G$ and $x \in \Sigma^*$ it is undecideable whether $x \in L(G)$
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- The *Halting Problem (HP)* is recursively enumerable so there is a grammar $G$ so that $L(G) = HP$
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- The *Halting Problem* (HP) is recursively enumerable so there is a grammar $G$ so that $L(G) = HP$
- We could construct a non-deterministic Turing Machine $M$ that simulates the rule applications of $G$ . . .
Context-sensitive Grammars

Definition

A context-sensitive grammar is a grammar for which whenever $(x, y) \in R$ we have $|x| \leq |y|$
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\((x, y) \in R\) we have \(|x| \leq |y|\)

Example

There is a context-sensitive grammar that generates the language 
\(L = \{xx : x \in \Sigma^*\}\)

\[
\begin{align*}
S & \rightarrow a_ia_i \\
S & \rightarrow a_iA_i \\
S & \rightarrow a_iSa_i \\
S & \rightarrow a_iSA_i \\
A_ia_j & \rightarrow a_jA_i \\
A_i\epsilon & \rightarrow a_i\epsilon
\end{align*}
\]

where \(a_i, a_j \in \Sigma\) and \(A_i \in N \setminus \{S\}\)
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Given grammar $G$ and $x \in \Sigma^*$ it is decideable whether $x \in L(G)$
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1. $w := S$, $\text{Store} := \emptyset$

2. Choose $y \notin \text{Store}$ such that $w \rightarrow^1 y$
   
   if $|x| \leq |y|$ then $\text{halt}$ with “no”
   
   else if $y = x$ then $\text{halt}$ with “yes”
   
   else $\text{Store} := \text{Store} \cup \{w\}$, $w := y$, repeat
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Due to the restriction $(x, y) \implies |x| \leq |y|$ we need at most $\sum_{i=0}^{\frac{|x|}{\|\Sigma\|}} |\Sigma|^i$ steps to surpass the length of $x$
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The class of languages generated by context-sensitive grammars is precisely $\text{NSPACE}(n)$

- There is a non-deterministic algorithm that decides $L(G)$ with additional space $n$
  
  1. Choose $x_1, x_2, \ldots, x_k$ so that
     
     $S \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_k$ and
     
     $|x_i| \leq |x|

  2. if $x_k \equiv x$ then $halt$ with “yes” else $halt$ with “no”
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     $|x_i| \leq |x|
  2. If $x_k \equiv x$ then *halt* with “yes” else *halt* with “no”

- If a non-deterministic Turing Machine using additional space $n$ decides $L$ then there is a context-sensitive $G$ such that $L = L(G)$
  - The string representation of the machine’s configuration has length $n + 3$ at most
  - We can use $\square$ (blank) as a terminal symbol of $G$
  - We can design the grammar rules so that the string being manipulated has always length $n + 3$ (padding with $\square$)
  - But this is a context-sensitive grammar...
Definition

A grammar is context-free if, for all rules \((x, y) \in R, x \in N\)
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Example
There is a context-free grammar that generates the language of balanced parentheses

\[
S \rightarrow SS \\
S \rightarrow (S) \\
S \rightarrow () \\
S \rightarrow \epsilon
\]
Lemma

Given grammar $G$ and $x \in \Sigma^*$ it is in $P$ to decide whether $x \in L(G)$.
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Papadimitriou gives a dynamic-programming algorithm which solves this problem in polynomial time.
Right-linear Context-free Grammars

Definition

A context-free grammar is right-linear if \( R \subseteq N \times (\Sigma N \cup \{\epsilon\}) \)
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Example
There is right-linear context-free languages that generates all strings of 1, 0 that end in 101

\[
egin{align*}
S & \rightarrow 0S \\
S & \rightarrow 1S \\
S & \rightarrow 1A \\
A & \rightarrow 0B \\
B & \rightarrow 1C \\
C & \rightarrow \epsilon
\end{align*}
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- For every such grammar $G$ we can construct a NFA that accepts $L(G)$
  1. For every non-terminal symbol of $G$ we create a new state for the NFA
  2. For every rule $A \rightarrow aB$ we create a transition $A_s \xrightarrow{a} B_s$
  3. For every rule $A \rightarrow \epsilon$ we define state $A_s$ to be an accepting state

- For every language $L$ accepted by a DFA we can construct a right-linear context-free grammar $G$ such that $L = L(G)$
  1. For every transition $q \xrightarrow{a} p$ we create a rule $Q \rightarrow aP$ (we add $Q$, $P$ in $N$ if they are not already there)
  2. For every accepting state $q$ we create a rule $Q \rightarrow \epsilon$ (we add $Q$ in $N$ if it is not already there)
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