An overview of approximation algorithms for the Distance Constraint Vehicle Routing

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Topics covered in the presentation

- Definition and properties of Distance Constraint Vehicle Routing Problem **DVRP**
- A 3—approximation algorithm for the unrooted DVRP by [1]
- Bicriteria approximation algorithm by [2]
Definition of DVRP

Input

- A complete graph $G = (V, E)$
- A metric distance $d : E \rightarrow \mathbb{R}^+$
- A starting position (depot, root) $r$
- A bound on the allowed length of a tour $D$

Output

A set of tours starting from $r$, with length at most $D$ with the minimum cardinality ($C$), for which all vertices belong to at least one tour.

Definition: Unrooted DVRP (or minimum path cover) is a DVRP where the goal is to find the minimum cardinality set of paths (i.e. start and end location of every route is not the same) covering all vertices.
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Properties of DVRP

**DVRP is NP-hard**
Decision TSP (DTSP) can be reduced to the decision version of DVRP, where there exists a tour covering $V$ with length at most $D$ if and only if there exists a set of tours from $r$ with length at most $D$ with cardinality at most 1.

**It is hard to approximate DVRP within a factor of 2**
If there is a polynomial $a$–approximation algorithm $A$ for the DVRP, $a < 2$, then the DTSP can be answered in polynomial time, since there exists a tour covering $V$ with length at most $D$ if and only if the output of $A$ is at most $a$.
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A 3—approximation algorithm for the unrooted DVRP [1], Intuitions

- The minimum cardinality \( k \) is guessed. This can be done, since possible results are \( 1, 2, \ldots, n = |V| \) (polynomially bounded by the input), so exhaustive search can be applied.

- If \( t_1, t_2, \ldots, t_k \) constitute a solution to DVRP, then these constitute \( k \) connected components of \( G \). So, the minimum \( k \) connected components \( C_i, i = 1, 2, \ldots, k \) have
  \[
  \sum l(C_i) \leq \sum l(t_i) \leq k \cdot D
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- By doubling each edge in the connected components \( k \) Eulerian paths \( p_1, p_2, \ldots, p_k \) can be created. They have total length
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- Each \( p_i \) can be cut into subpaths \( p_{i1}, p_{i2}, \ldots, p_{il_i} \), where
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  with length at most \( D \) and
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  l_1 + l_2 + \cdots + l_k \leq 3 \cdot k,
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  since
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A 3—approximation algorithm for the unrooted DVRP [1], Sketch of the algorithm

For every possible minimum cardinality ($k = 1, 2, \ldots, n$)

1. Compute the $k$ minimum connected components $C_1, C_2, \ldots, C_k$ (using the Kruskal’s algorithm for minimum spanning tree)

2. For each component $C_i$ double its edges and compute an eulerian path $p_i$

3. Cut each $p_i, i = 1, 2, \ldots, k$ into segments $p_i1, p_i2, \ldots p_il_i$ that have length at most $D$ with $p_i = p_i1 * p_i2 * \cdots p_il_i$. Call

$$S_i^k = \{p_i1, p_i2, \ldots p_il_i\}$$

4. $S_k = \cup S_i^k$

Return the $S_k$ with the minimum cardinality
Bicriteria approximation algorithm for DVRP [2]

**Theorem**: There is a $O(\log \frac{1}{\epsilon}, 1 + \epsilon)$ bicriteria approximation algorithm for DVRP. (For $0 < \epsilon < 1$ if each tour is allowed to have length at most $(1 + \epsilon) \cdot D$, then a set of tours containing $V$ with cardinality at most $\log \frac{1}{\epsilon}$ times an optimal solution can be found.)
Bicriteria approximation algorithm for DVRP [2], Intuitions

- The set of vertices $V$, is partitioned into $1 + \lceil \log \frac{1}{\epsilon} \rceil$ subsets $V_0, V_1, \ldots, V_{\lceil \log \frac{1}{\epsilon} \rceil}$ according to their distance from the depot. Specifically,
  
  $V_0 = \{ v : (1 - \epsilon) \frac{D}{2} < d(r, v) \leq \frac{D}{2} \}$ and
  $V_j = \{ v : (1 - 2^j \epsilon) \frac{D}{2} < d(r, v) \leq (1 - 2^{j-1} \epsilon) \frac{D}{2} \}$, $j = 1, 2, \ldots, \lceil \log \frac{1}{\epsilon} \rceil$

- If there is a path $P(v_1, v_2, \ldots, v_k) \subseteq V_j$ with $l(P) \leq 2^{j-1} \epsilon D$ then the tour $r \ast P \ast r = (r, v_1, v_2, \ldots, v_k, r)$ has length at most $d(r, v_1) + l(P) + d(v_k, r) \leq (1 + \epsilon) \cdot D$.

- Let a tour $t = (r, u_1, u_2, \ldots, u_k, r)$ belonging to a solution of DVRP, then the restriction of $t$ in $V_j$, $t_{V_j} = (u_{m_1}, u_{m_2}, \ldots, u_{m_l}) \subseteq V_j$, $m_1 < m_2 < \cdots < m_l$ has length less than $2^j \cdot \epsilon \cdot D$. Furthermore, $t_{V_j}$ can be cut into two paths with length less than $2^{j-1} \cdot \epsilon \cdot D$. So, there are at most $2 \cdot \text{OPT}$ paths covering $V_j$ bounded by $2^{j-1} \cdot \epsilon \cdot D$.

- For each $j = 0, 1, \ldots, \lceil \log \frac{1}{\epsilon} \rceil$, at most $6 \cdot \text{OPT}$ paths covering $V_j$ can be found, with length at most $2^{j-1} \cdot \epsilon \cdot D$, applying the $3$–approximation algorithm described. So, at most $6 \cdot \text{OPT} \cdot (1 + \lceil \log \frac{1}{\epsilon} \rceil)$ tours bounded by $(1 + \epsilon) \cdot D$ covering $V$ are created.
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Bicriteria approximation algorithm for DVRP [2], Sketch of the algorithm

1. Partition $V$ in $V_0, V_1, \ldots, V_{\lceil \log \frac{1}{\epsilon} \rceil}$

2. For each $V_j$ calculate a set of paths $P_j$ in it, bounded by $2^{j-1} \cdot \epsilon \cdot D$ using the 3-approximation algorithm described above

3. Take $P = \cup P_j$

4. Then the set of tours is $T = r \ast P \ast r = \{r \ast p \ast r \mid p \in P\}$
E. M. Arkin, R. Hassin, and A. Levin.  
Approximations for minimum and min-max vehicle routing problems.  

V. Nagarajan and R. Ravi.  
Approximation algorithms for distance constrained vehicle routing problems.  