Coloring Circular Arc Graphs

Revisiting Tucker’s algorithm

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Outline

Introduction

Analyzing Tucker’s algorithm
The problem

Input: a family $F$ of circular arcs

Output: is there a proper coloring with $\leq k$ colors?

what is the minimum $k$ s.t. $F$ has a proper coloring?
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Some quantities

\[ L(F) = 3 \]

\[ l(F) = 4 \]

\[ \omega(F) = \text{Load of } N \text{ circular-cover: } l \]

\[ N \text{ max. clique of } F: \omega \text{(as usual)} \]

We also discretize and use the -at most- \(|F|\) points defining the arcs.
Some quantities

Load of $F$: $L$
Some quantities

- Load of $F$: $L$
- Circular-cover: $l$
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- Load of $F$: $L$
- circular-cover: $l$
- max. clique of $F$: $\omega$
  (as usual)
- We also discretize and use the -at most- $2|F|$ points defining the arcs.
Results

Theorem 1 (Tucker (1975)).

Let $F$ be a family of circular arcs with load $L = L(F)$ and circular-cover $l = l(F)$. If $l(F) \geq 4$, then $\lfloor \frac{3}{2} L \rfloor$ colors suffice to properly color $F$.

- This is actually a **2-approximation** algorithm.
  - Tucker, [4] conjecture that $\chi(F) \leq \frac{3}{2} \omega(F)$.
  - Karapetian (1980) proves the above.
  - Garey et al. (1980) show NP-completeness for CIRCULAR ARC COLOR
  - Many exact algos for subfamilies of graphs ($\geq O(|F|^{1.5})$).
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More recently:

**Theorem 2 (Valencia-Pabon (2003)).**

Consider $F$ with load $L(F)$ and circular-cover $l(F) \geq 5$. Then
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\left\lceil \frac{l-1}{l-2} L \right\rceil
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That's this presentation about!

It is based exactly on the algorithm proposed by Tucker, [4].
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Introduction

Analyzing Tucker’s algorithm
Tucker’s algorithm

Select $p$ so that $L(F)$ arcs contain it.

Assign color #1 to the arc which extends at least on the counterclockwise side of $p$.

Move clockwise, assign current color to the first arc to begin after the previous ends.

Unless it is not possible, so use next color.
Tucker’s algorithm

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Notations needed

Starting point: \( t \) first arc colored with \( A_i \):

Arcs colored until \( k \)th round: \( F_k \)
Notations needed

- starting point: $t$

Coloring Circular Arc Graphs • Analyzing Tucker’s algorithm
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- starting point: $t$
- first arc colored with $i$: $A_i$
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Properties

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2. $L(F \setminus F_i) \leq L(F) - i$
   OK, verify again later*
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   Proof?

Base of induction
$1 \leq i \leq l - 2$

Suppose $F_{i-1}$ is properly colored with $i$ colors. Let:
Proving Property 3

Suppose \( F_{i-1} \) is properly colored with \( i \) colors. Let:

- \( A_i \) the first to get color \( i \) in round \( i - 1 \)
- \( A_i^f \) the last of round \( i - 1 \) to traverse \( t \)

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In this case:
\( A_2, A_3, \ldots, A_i, A_i^f \) cover the circle!
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Recap

**Property 2.** \( L(F \setminus F_{l-2}) \leq L - (l - 2) \)

**Property 3.** \( F_{l-2} \) are properly colored with \( l - 1 \) colors.

But how to use some induction here?

Maintain little arcs \((p, p + 1)\) to create a constant load of \( L(F) \) around the circle. Neither \( \chi(F) \) nor the Algorithm’s output is changed! *Now check again Property 2!*

Now: \( F' = F \setminus F_{l-2} \) has \( l_{F'} \geq l_F \geq 5 \), so use induction!...

- \( L \) rounds, every \( l - 2 \) rounds need at most \( l - 1 \) colors
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Some final interaction

◆ What if every arc in $F$ spans at most $n/k$ “points” of the circle?

$$l \geq k + 1 \Rightarrow SOL = \left\lceil \frac{k}{k-1} L \right\rceil$$

◆ What if every arc spans more than a semi-circle?

The circular arc graph is complete!

◆ To show tightness, Valencia-Pabon uses a result of Stahl, [3] regarding

$r$-tuple colorings...

Major open problem: better than $\frac{3}{2}$-approximation?
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That's all folks!

Thank you!


