Approximation Algorithms

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Outline

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2. Vertex Cover
3. Knapsack
4. TSP

1. Introduction

Optimization Problems

- Optimization Problem: Every instance of the problem corresponds to some feasible solutions each of them having a value via an Objective Function.
- We seek for an Optimal Solution i.e. a feasible solution that has an optimal value.
- Optimization problems can be either Maximization or Minimization
- Example: The Vertex Cover Problem
  - Min or Max: Mimimization
  - Instance: A graph
  - Feasible Solutions: Every Vertex Cover
  - Objective Function: The cardinality $|\star|$ function
  - Optimal Solution: A Vertex Cover of minimum cardinality

The PO-class

We call the class of optimization problems that can be optimally solved in polynomial time PO class (PO stands for P-Optimization).

Examples: SHORTEST PATH, MAXIMUM MATCHING, ...

Relation of P to PO (i)

Consider a minimization problem Π such that the size (in bits) of a feasible solution is polynomial in the size of the input. Assume also that the objective function is efficiently computable.

Π: *given an instance of size n find a feasible solution of minimum value.*

The corresponding decision problem is:

Π_d: *Given an instance of Π of size n and an integer k is there a feasible solution of value less or equal to k?*

$\rightsquigarrow$ If the decision version is polynomially solvable on n and $\log k$ then we can construct a polynomial time algorithm for the optimization version (in most cases)

Relation of P to PO (ii)

- Determine bounds A, B, such that any feasible solution is of value between A and B.
- Then do binary search in $[A, B]$ to find the optimum value k ($\log(B - A)$ runs of the decision version algorithm).
- Exploit knowledge of k in order to determine the optimum solution (not known how to do this in general).

The above (if everything works) is a polynomial time algorithm in the size of the input. Therefore, Π lies in PO.

The NPO-class: NP-Optimization Problems

- Each instance is associated with at least one feasible solution.
- The size (in bits) of any feasible solution is bounded by a polynomial in the input size (n).
- The objective function is in class FP, i.e. it is poly-time computable in the size of a feasible solution (hence also in n).

Relation to NP: the decision version of an NPO problem is in NP. Several NP-complete decision problems correspond to problems in NPO which are consequently NP-hard (*why?*).
What can we do then?

- Solve the problem exactly on limited instances.
- Find polynomial time approximation algorithms

Notation

- Π: Problem
- I: Instance
- $\text{SOL}_A(\Pi, I)$: The solution we obtain for the instance I of the problem Π using algorithm A.
- $\text{OPT}(\Pi, I)$: The optimal solution for the instance I of the problem Π.

Note: We usually omit Π, I and A from the above notation.

Approximability

- An algorithm A for a minimization problem Π achieves a ρ_A approximation factor, ($\rho_A : \mathbb{N} \to \mathbb{Q}^+$) if for every instance I of size $|I| = n$:

$$\frac{\text{SOL}_A(I)}{\text{OPT}(I)} \leq \rho_A(n)$$

- An algorithm A for a maximization problem Π achieves a ρ_A approximation factor, ($\rho_A : \mathbb{N} \to \mathbb{Q}^+$) if for every instance I of size $|I| = n$:

$$\frac{\text{SOL}_A(I)}{\text{OPT}(I)} \geq \rho_A(n)$$

$\rightsquigarrow$ An approximation algorithm of factor ρ guarantees that the solution that the algorithm computes cannot be worse than ρ times the optimal solution.

Approximation Schemes

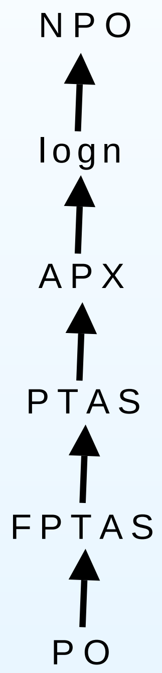
Informally: We can have as good approximation factor as we want trading off time.

Formally:

- A is an Approximation Scheme (AS) for problem Π if on input (I, ε), where I an instance and $\varepsilon > 0$ an error parameter:
  - $\text{SOL}_A(I, \varepsilon) \leq (1 + \varepsilon) \cdot \text{OPT}(I)$, for minimization problem
  - $\text{SOL}_A(I, \varepsilon) \geq (1 - \varepsilon) \cdot \text{OPT}(I)$, for maximization problem
- A is a PTAS (Polynomial Time AS) if for every fixed $\varepsilon > 0$ it runs in polynomial time in the size of I.
- A is an FPTAS (Fully PTAS) if for every fixed $\varepsilon > 0$ it runs in polynomial time in the size of I and in $1/\varepsilon$.

Approximation World

Depending on the approximation factor we have several classes of approximation:



- logn: $\rho(n) = O(\log n)$
- APX: $\rho(n) = \rho$ (constant factor approximation)

Representatives

- Non-approximable: Traveling Salesman Problem
- $\log n$: Set Cover
- APX: Vertex Cover / Ferry Cover
- PTAS: Makespan Scheduling
- FPTAS: Knapsack

2. Vertex Cover

The (Cardinality) Vertex Cover Problem

Definition: Given a graph $G(V, E)$ find a minimum cardinality Vertex Cover, i.e. a set $V' \subseteq V$ such that every edge has at least one endpoint in V'.

- A trivial feasible solution would be the set V
- Finding a minimum cardinality Vertex Cover is NP-hard (reduction from 3-SAT)
- An approximation algorithm of factor 2 will be presented

Lower Bounding

A general strategy for obtaining a ρ-approximation algorithm (for a minimization problem) is the following:

- Find a lower bound l of the optimal solution ($l \leq \text{OPT}$)
- Find a factor ρ such that $\text{SOL} = \rho \cdot l$

$\rightsquigarrow$ The previous scheme implies $\text{SOL} \leq \rho \cdot \text{OPT}$

Matchings

- Definition: Given a graph $G(V, E)$ a matching is a subset of the edges $M \subseteq E$ such that no two edges in M share an endpoint.
- Maximal Matching: A matching that no more edges can be added.
- Maximum Matching: A maximum cardinality matching.

$\rightsquigarrow$ Maximal Matching is solved in polynomial time with the greedy algorithm
$\rightsquigarrow$ Maximum Matching is also solved in polynomial time via a reduction to max-flow

A 2-Approximation Algorithm for Vertex Cover

- The Algorithm: Find a maximal matching M of the graph and output the set V' of matched vertices
- Correctness:
  - Edges belonging in M are all covered by V'
  - Since M is a maximal matching, any other edge $e \in E \setminus M$ will share at least one endpoint v with some $e' \in M$. So v is in V' and guards e.
- Analysis:
  - Any vertex cover should pick at least one endpoint of each matched edge $\rightarrow |M| \leq \text{OPT}$
  - $|V'| = 2|M|$

  Thus $\text{SOL} = |V'| = 2|M| \leq 2\text{OPT} \Rightarrow \text{SOL} \leq 2\text{OPT}$

$\rightsquigarrow$Vertex Cover is in APX

Can we do better?

Questions

- Can the approximation guarantee be improved by a better analysis?
- Can an approximation algorithm with a better guarantee be designed using the same lower bounding scheme?
- Is there some other lower bounding methods that can lead to an improved approximation algorithm?

Answers

- Tight Examples
- Other kind of examples
- This is not so immediate...

Tight Examples

- A better analysis might imply an l' s.t. $l < l' \leq \text{OPT}$. Then there would be a $\rho' < \rho$ s.t. $\rho \cdot l = \rho' \cdot l'$, so

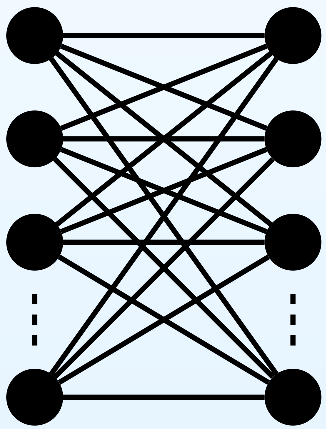
$$\text{SOL} = \rho \cdot l = \rho' \cdot l' \leq \rho' \text{OPT}$$

  Thus we could obtain a better approximation factor $\rho' < \rho$.

- Definition: An infinite family of instances in which $l = \text{OPT}$ is called Tight Example for the ρ-approximation algorithm.
- If $l = \text{OPT}$ then there is no $l' > l$ s.t $l' \leq \text{OPT}$.
  $\rightsquigarrow$So we can't find a better factor by better analysis

Tight Example for the matching algorithm

- The infinite family $K_{n,n}$ of the complete balanced bipartite graphs is a tight example.
- $|M| = n = \text{OPT}$. So the solution returned is 2 times the optimal solution.

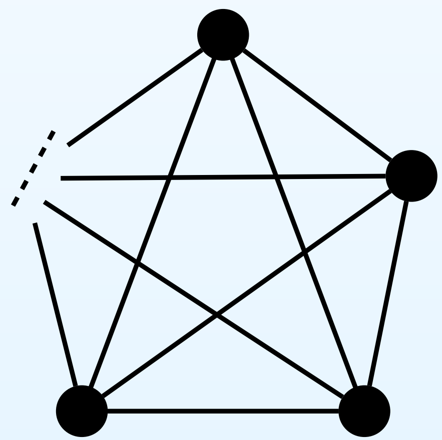


Other kind of examples

- Using the same lower bound $l \leq$ OPT we might find a better algorithm with $\rho' < \rho$ that computes SOL $= \rho' \cdot l$. This would imply a better ρ' approximation algorithm.
- An infinite family where $l = \frac{1}{\rho}$OPT implies that SOL $= l \cdot \rho' = \frac{1}{\rho}\rho'$OPT $<$ OPT (contradiction).
  $\rightsquigarrow$Thus it is impossible to find another algorithm with better approximation factor using the lower bound $l \leq$ OPT

Using the matching lower bound

- The infinite family K_{2n+1} of the complete bipartite graphs with odd number of vertices have an optimal vertex of cardinality $2n$
- A maximal matching could be $|M| = n = \frac{1}{2}\text{OPT}$. So the solution returned is the optimal solution.



Other lower bounds for Vertex Cover

- This is still an open research area.
- Best known result for the approximation factor (until 2004) is $2 - \Theta(\frac{1}{\sqrt{\log n}})$ (due to George Karakostas)
- Uses Linear Programming.

3. Knapsack

Pseudo-polynomial time algorithms

- An instance I of any problem Π consists of objects (sets, graphs,...) and numbers.
- The size of I ($|I|$) is the number of bits needed to write the instance I.
- Numbers in I are written in binary
- Let I_u be the instance I where all numbers are written in unary
- Definition: A pseudo-polynomial time algorithm is an algorithm running in polynomial time in $|I_u|$
- Pseudo-polynomial time algorithms can be obtained using Dynamic Programming

Strong NP-hardness

- Definition: A problem is called strongly NP-hard if any problem in NP can be polynomially reduced to it and numbers in the reduced instance are written in unary
- Informally: A strongly NP-hard problem remains NP-hard even if the input numbers are less than some polynomial of the size of the objects.

$\rightsquigarrow$ Strongly NP-hard problems cannot admit a pseudo-polynomial time algorithm, assuming $P \neq NP$

(else we could solve the reduced instance in polynomial time, thus we could solve every problem in NP in polynomial time. That would imply $P = NP$)

The existence of FPTAS

Theorem: For a minimization problem Π if $\forall$ instance I,

- OPT is strictly bounded by a polynomial of $|I_u|$ and
- the objective function is integer valued

then Π admits an FPTAS $\Rightarrow$ Π admits a pseudo-polynomial time algorithm

$\rightsquigarrow$ A strongly NP-hard problem (under the previous assumptions) cannot admit an FPTAS unless $P = NP$

The Knapsack Problem (i)

- Definition: The discrete version is given a set of n items $X = \{x_1, \ldots, x_n\}$ where a $profit : X \to \mathbb{N}$ and a $weight : X \to \mathbb{N}$ function are provided and a “knapsack” of total capacity $B \in \mathbb{N}$, find a subset $Y \subseteq X$ whose total size is bounded by B and maximizes the total profit.
- Definition: The continuous version is given a set of n continuous items $X = \{x_1, \ldots, x_n\}$ where $profit$ and $weight$ function are provided and a “knapsack” of total capacity $B \in \mathbb{N}$, find a sequence $\{w_1, \ldots, w_n\}$ of portions where $\sum_{i=1}^{n} w_i = B$ that maximizes the total profit.

The Knapsack Problem (ii)

- The greedy algorithm (sort the objects by decreasing ratio of profit to weight) solves in polynomial time the continuous version
- The greedy algorithm can be made to perform arbitrarily bad for the discrete version.
- Discrete Knapsack is NP-hard
- Pseudo-polynomial time and FPTAS algorithms will be presented for the discrete version.
- For now on we focus on discrete knapsack and call it "knapsack"

A pseudo-polynomial time algorithm for knapsack (i)

- Let P be the profit of the most profitable object
- nP is a trivial upper bound on the total profit
- For $i \in \{1, \ldots, n\}$ and $p \in \{1, \ldots, nP\}$ let $S(i, p)$ denote a subset of $\{x_1, \ldots, x_i\}$ whose total profit is exactly p and its total weight is minimized
- Let $W(i, p)$ denote the weight of $S(i, p)$ (∞ if no such a set exists)

A pseudo-polynomial time algorithm for knapsack (ii)

The following recursive scheme computes all values $W(i,p)$ in $O(n^2P)$

- $W(1,p) = weight(x_1)$, if $p = profit(x_1)$, ∞ else
- $W(i+1,p) =$
$$\begin{cases} W(i,p), & \text{if } profit(x_{i+1}) > p \\ \min\{W(i,p), weight(x_{i+1}) + W(i, p - profit(x_{i+1}))\}, & \text{else} \end{cases}$$

The optimal solution of the problem is $\max\{p \mid W(n,p) \leq B\}$

$\rightsquigarrow$The optimal solution can be computed in polynomial time on n and P

An FPTAS for Knapsack

- Idea: The previous algorithm could be a polynomial time algorithm if P was bounded by a polynomial of n
- Ignore a number of least significant bits of the profits of the objects
- Modified profits $profit'$ should now be numbers bounded by a polynomial of n and $\frac{1}{\varepsilon}$ (ε is the error parameter)
- The algorithm:
  1. Given $\varepsilon > 0$ define $K = \frac{\varepsilon P}{n}$
  2. Set new profit function $profit'$, $profit'(x_i) = \lceil \frac{profit(x_i)}{K} \rceil$
  3. Run the pseudo-polynomial time algorithm described previously and output the result

Analysis

Theorem: The previous algorithm is an FPTAS

1. SOL $\geq (1-\varepsilon)$OPT
2. Runs in polynomial time in n and $\frac{1}{\varepsilon}$

Proof:

1. Let S and O denote the output set and the optimal set
   - $profit'(x_i) = \lceil \frac{profit(x_i)}{K} \rceil \Rightarrow$
     $profit(x_i) \leq K \cdot profit'(x_i) \leq profit(x_i) + K$
   - $\forall A \subseteq X: \ profit(A) \leq K \cdot profit'(A) \leq profit(A) + n \cdot K$
   - $K = \frac{\varepsilon P}{n}$, $\ profit'(S) \geq profit'(O)$, OPT $\geq P$

   Thus, SOL $= profit(S) \geq K \cdot profit'(S) - nK \geq$
   $K \cdot profit'(O) - nK \geq profit(O) - nK =$ OPT $- \varepsilon P$
   $\geq (1-\varepsilon) \cdot$ OPT
2. The algorithm's running time is $O(n^2 \lceil \frac{P}{K} \rceil) = O(n^2 \lceil \frac{n}{\varepsilon} \rceil)$

4. TSP

Hardness of Approximation

To show that an optimization problem Π is hard to approximate we can use

- A Gap-introducing reduction: Reduces an NP-complete decision problem Π' to Π
- A Gap-preserving reduction: Reduces a hard to approximate optimization problem Π' to Π

Gap-introducing reductions (i)

Suppose that Π' is a decision problem and Π a minimization problem (similar for maximization).
A reduction h from Π' to Π is called gap-introducing if:

1. Transforms (in polynomial time) any instance I' of Π' to an instance $I = h(I')$ of Π
2. There are functions f and α s.t.
    - If I' is a 'yes instance' of Π' then $\text{OPT}(\Pi, I) \leq f(I)$
    - If I' is a 'no instance' of Π' then $\text{OPT}(\Pi, I) > \alpha(|I|) \cdot f(I)$

Gap-introducing reductions (ii)

Theorem: If Π' is NP-complete then Π cannot be approximated with a factor α

Proof: If Π had an approximation algorithm of factor α then $\text{SOL} \leq \alpha \cdot \text{OPT}$. So,

- I' is a 'yes instance' of $\Pi' \Rightarrow \text{SOL} \leq \alpha \cdot \text{OPT}(\Pi, I) \leq \alpha \cdot f(I)$
- I' is a 'no instance' of $\Pi' \Rightarrow \text{SOL} > \text{OPT}(\Pi, I) > \alpha(|I|) \cdot f(I)$

Then by using the approximation algorithm for Π we could be able to determine in polynomial time whether the instance I' is 'yes' or 'no'.

Since Π is NP-complete, this would imply $P = NP$

Gap-preserving reductions (i)

Suppose that Π' is a minimization problem and Π a minimization (similar for other cases).

A reduction h from Π' to Π is called gap-preserving if:

1. Transforms (in polynomial time) any instance I' of Π' to an instance $I = h(I')$ of Π
2. There are functions f,f',α,β s.t.
   - OPT$(\Pi', I') \leq f'(I')$ $\Rightarrow$ OPT$(\Pi, I) \leq f(I)$
   - OPT$(\Pi', I') > \beta(|I'|) \cdot f'(I')$ $\Rightarrow$ OPT$(\Pi, I) > \alpha(|I|) \cdot f(I)$

Gap-preserving reductions (ii)

Theorem: If Π' is non-approximable with a factor β then Π cannot be approximated with a factor α unless $P = NP$

Proof: If Π had an approximation algorithm of factor α then SOL $\geq \alpha \cdot$ OPT. So,

- OPT$(\Pi', I') \leq f'(I') \Rightarrow$ SOL $\leq \alpha \cdot$ OPT$(\Pi, I) \leq \alpha \cdot f(I)$
- OPT$(\Pi', I') > \beta(|I'|) f'(I') \Rightarrow$ SOL $>$ OPT$(\Pi, I) > \alpha(|I|) \cdot f(I)$

But Π' cannot be approximated with a factor β means that there is an NP-complete decision problem Π'' and a gap-introducing reduction from Π'' to Π' s.t.

- I'' is a 'yes instance' of $\Pi'' \Rightarrow$ OPT$(\Pi', I') \leq f''(I')$
- I'' is a 'no instance' of $\Pi'' \Rightarrow$ OPT$(\Pi', I') > \beta(|I'|) \cdot f''(I')$

Thus, by running the algorithm for Π we could decide Π''. This implies $P = NP$

The Traveling Salesman Problem

Definition: Given a complete graph $K_n(V, E)$ and a weight function $w : E \to \mathbb{Q}$ find a tour, i.e. a permutation of the vertices, that has minimum total weight.

- The TSP problem is NP-hard
- TSP is non-approximable with a factor $\alpha(n)$ polynomial in n, via a gap-introducing reduction from Hamilton Cycle.

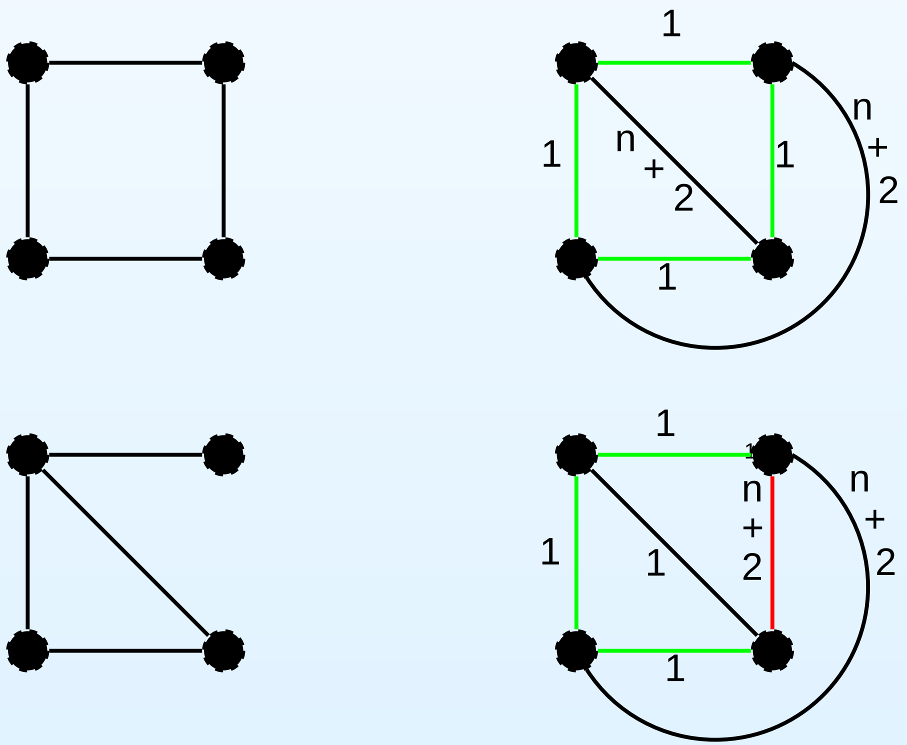
Definition: Given a graph $G(V, E)$ a Hamilton Cycle is a cycle that uses every vertex only ones.

- To determine whether G has a Hamilton Cycle or not is NP-complete.

TSP is non-approximable (i)

Reduction: $G(V, E), |V| = n$, is an instance of Hamilton Cycle. The instance of TSP will be K_n with a weight function w, $w(e) = 1$ if $e \in E$ else $w(e) = n + 2$. Then

- If G has a Hamilton Cycle then $\text{OPT}(\text{TSP}) = n$
- If I' is a ‘no instance’ of Π' then $\text{OPT}(\text{TSP}) > 2n$



TSP is non-approximable (ii)

- TSP is APX-hard, i.e there exist a constant α (in the example 2) that TSP cannot be approximated with factor α, unless $P = NP$
- Bonus!!! In the reduction if we set $w(e) = \alpha(n) \cdot n, e \notin E$ then we cannot have an $\alpha(n)$ approximation factor for TSP. Thus TSP is non-approximable

THE END!!!