Maximum Independent Set of Rectangles

Joint work with Orestis Konstantinides as part of the graduate course "Approximation Algorithms"

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Outline

Introduction

The challenge

The algorithm

In the loop, step 1: initial partition

In the loop, step 2: removal of bad rectangles

In the loop, step 3: removal of large cliques

Epilogue

The problem

Input: A set $\mathcal{R}$ of axis-parallel rectangles

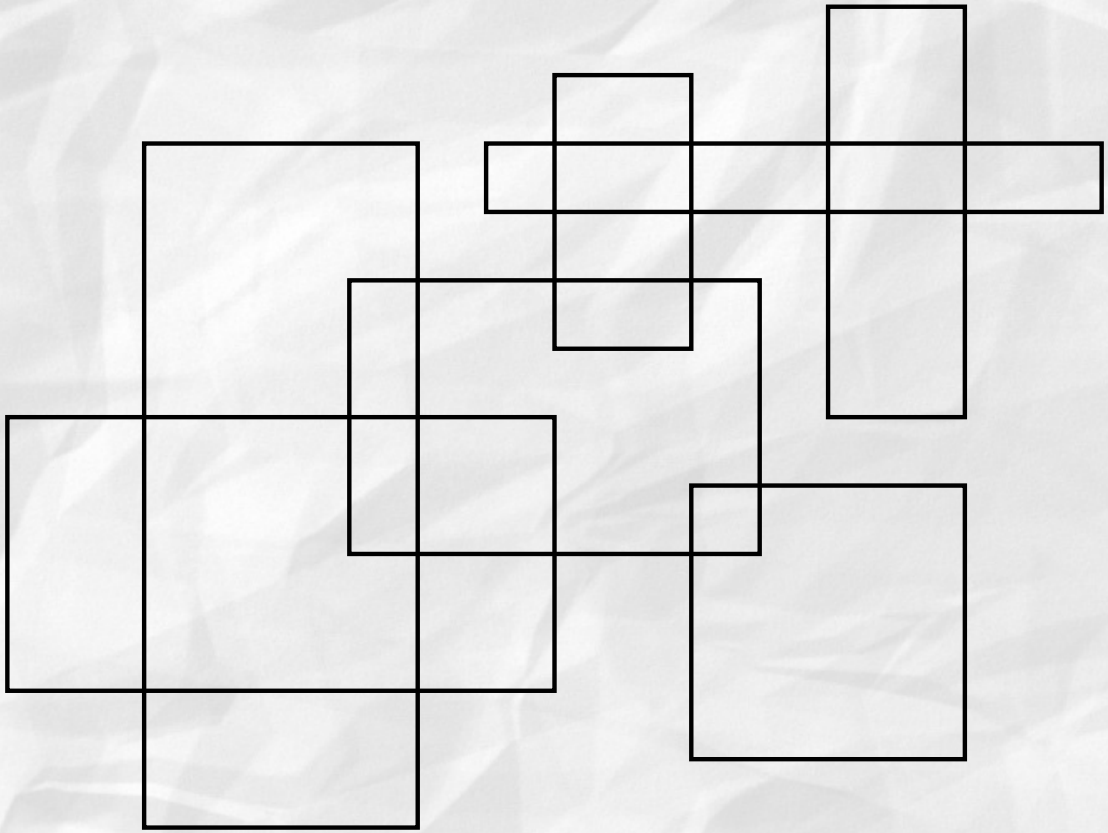

Figure: Instance of MISR

The problem

Input: A set $\mathcal{R}$ of axis-parallel rectangles
Output: The max. independent set of $\mathcal{R}$

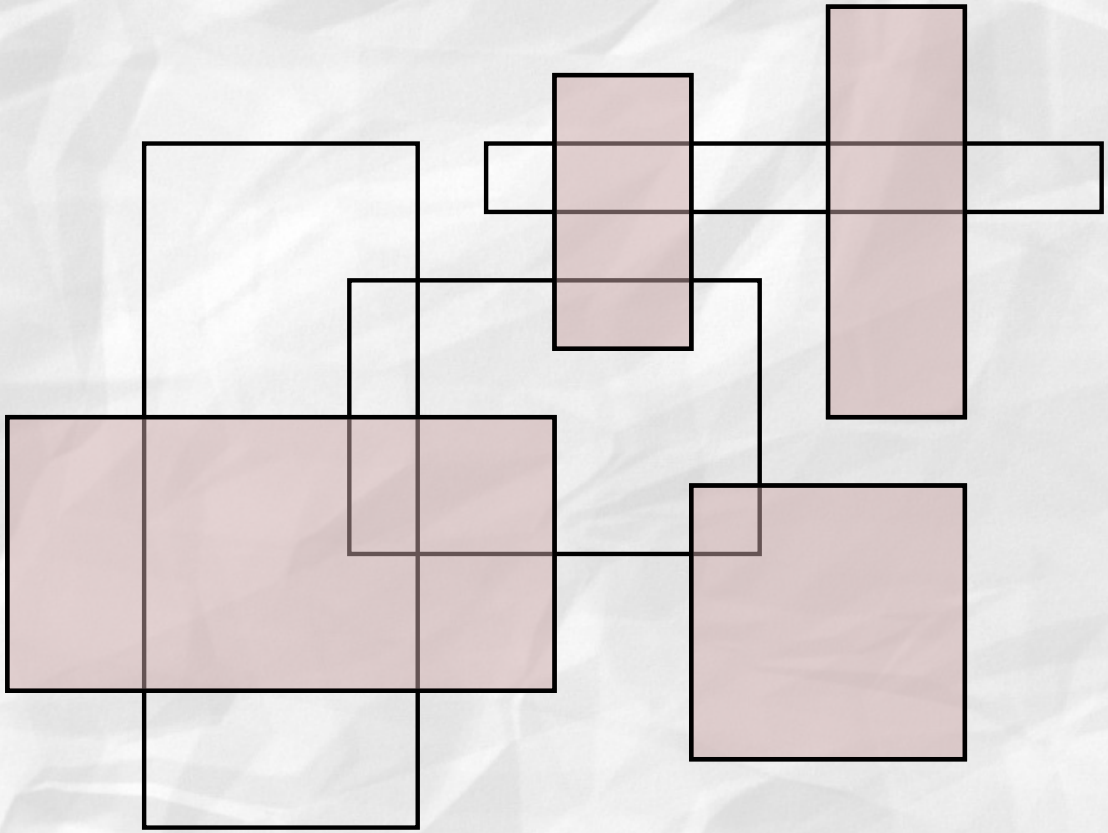

Figure: Instance of `MISR`

Existing results

- Agarwal et al. (1997), Khanna et al. (1998), Nielsen (2000): $\log n$-**approx. for** MISR
  - Agarwal et al. (1997): **PTAS** for rectangles with unit heights (improved running time by Chan (2004)
  - Chan (2003): **PTAS** for squares
  - Erlebach et al. (2005): **PTAS** for squares and rectangles with bounded aspect ratio
  - Lingas and Wahlen (2005): exact, in time $2^{\mathcal{O}\sqrt{n}\log n}$

Plus:

Theorem 1 (Lewin-Eytan et al. (2002)).

There is a parametrized $4q$-approximation for MISR *q being the cardinality of the maximum clique of the instance $\mathcal{R}$.*

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Theorem (Chalermsook and Chuzhoy, 2009).

There is a randomized $\log \log n$*-approximation algorithm for* MISR

Basic notations:

- Instance: $\mathcal{R}$, a rectangle: R
- For the LP: z_R indicating the **participation** or not of R in the solution

Basic principles:

- Solve LP relaxation and form a canonical solution
- Design an iterative algorithm defining partitions of $\mathcal{R}$

Fun fact:

- MAXIMUM CLIQUE for a set of rectangles is in **P**

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LP (and relaxation)

Our LP is the following:

$$\max \sum_{R \in \mathcal{R}} z_R$$

$$\text{s.t.} \sum_{R : p \in \mathcal{R}} z_R \leq 1, \quad \forall p \in \mathcal{P}$$

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- at most $(2n)^2$ points defined by our rectangles

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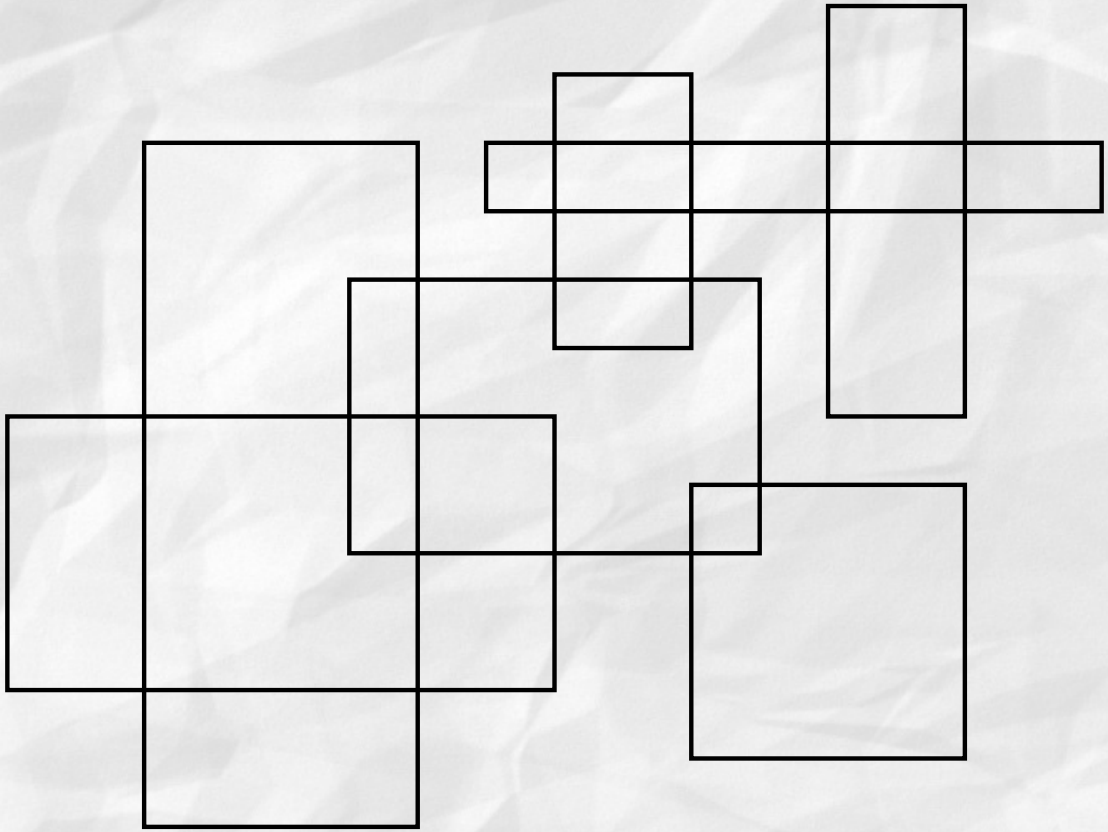

Figure: Set $\mathcal{P}$

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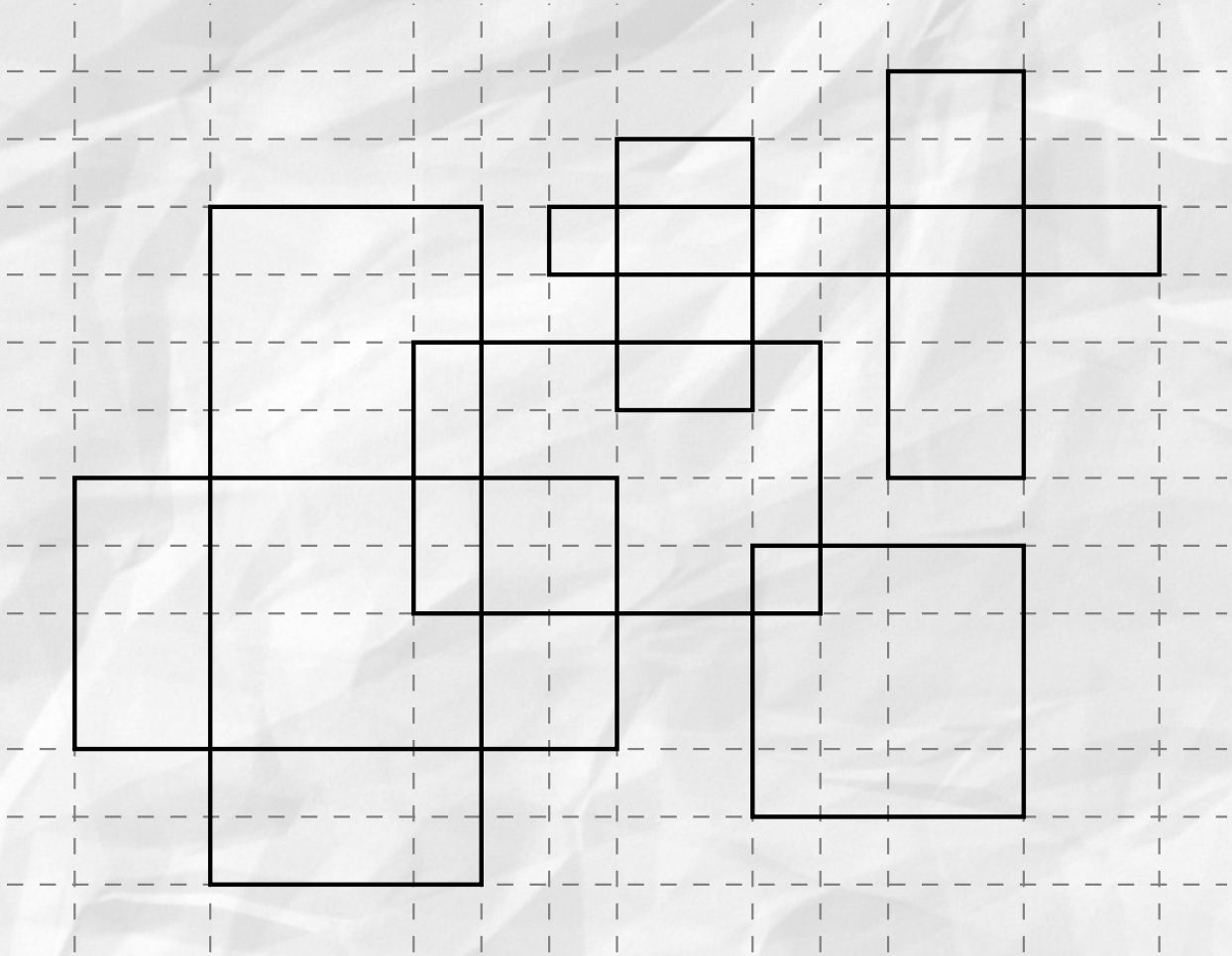

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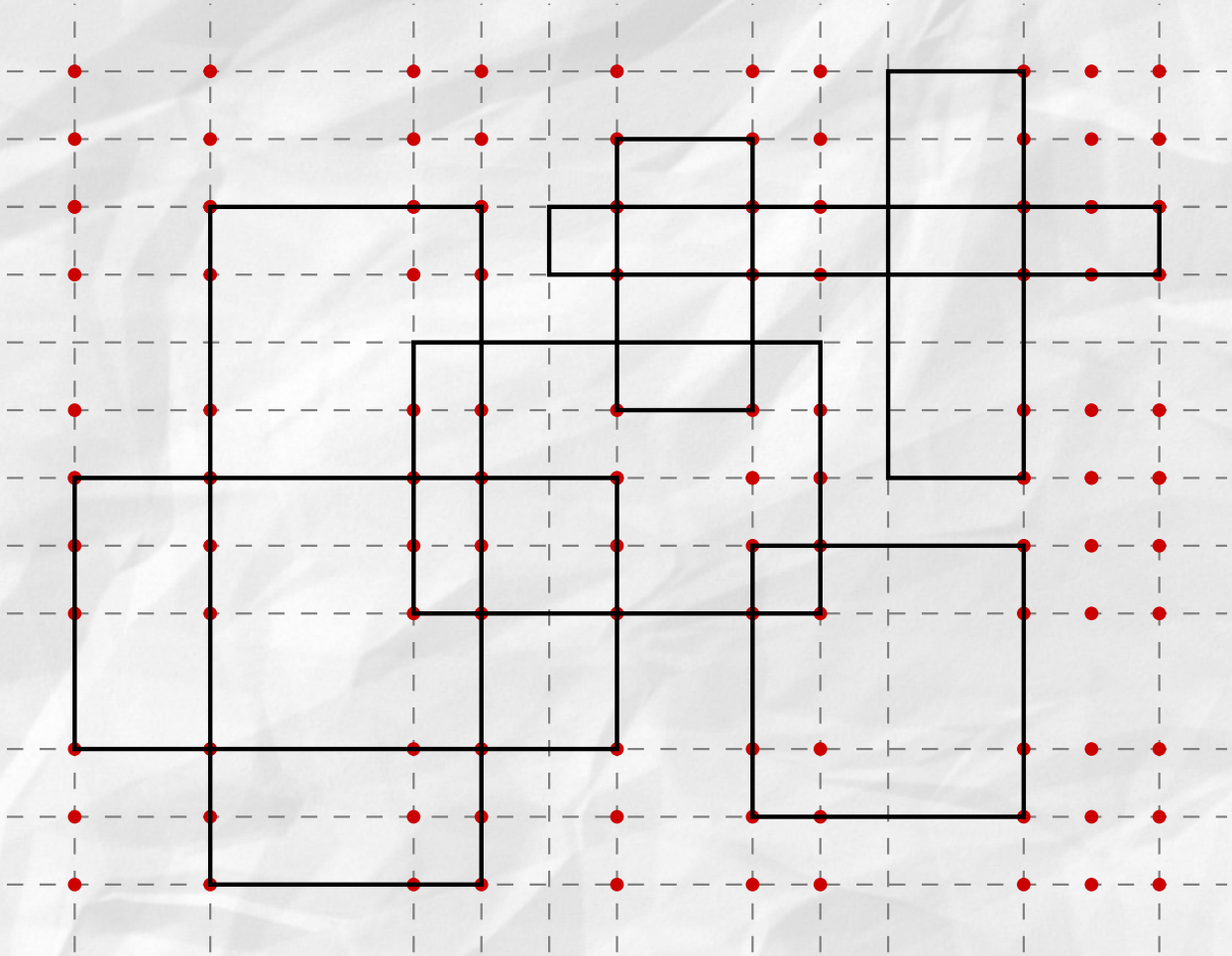

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Lemma 2.

There is an efficient randomized algorithm that, given an optimal LP solution of cost OPT for $\mathcal{R}$, produces w.h.p. a feasible canonical solution $\mathcal{R}'$ taking $z_R = \{0, \frac{1}{M}\}$, $M = \Theta(\log n)$ with cost OPT' $\geq c \cdot$ OPT.

Immediate consequences: largest clique in $\mathcal{R}'$ is at most M

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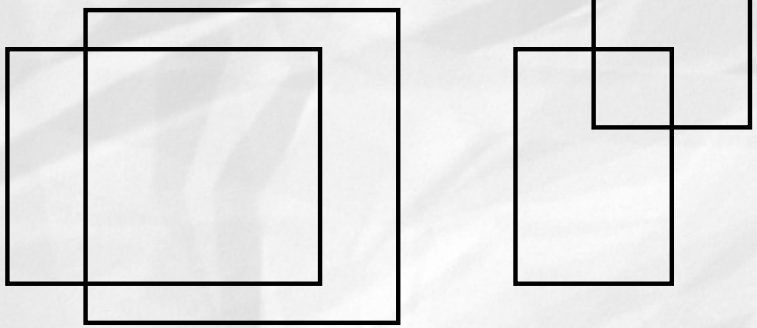
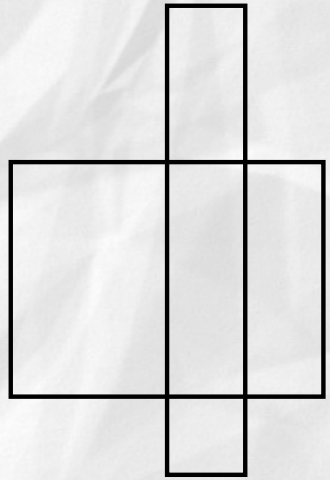
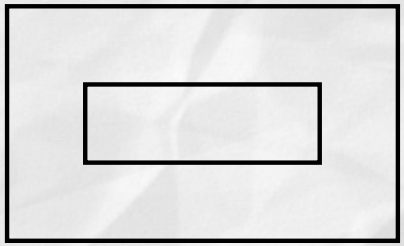
Types of intersection



(a) Corner intersections



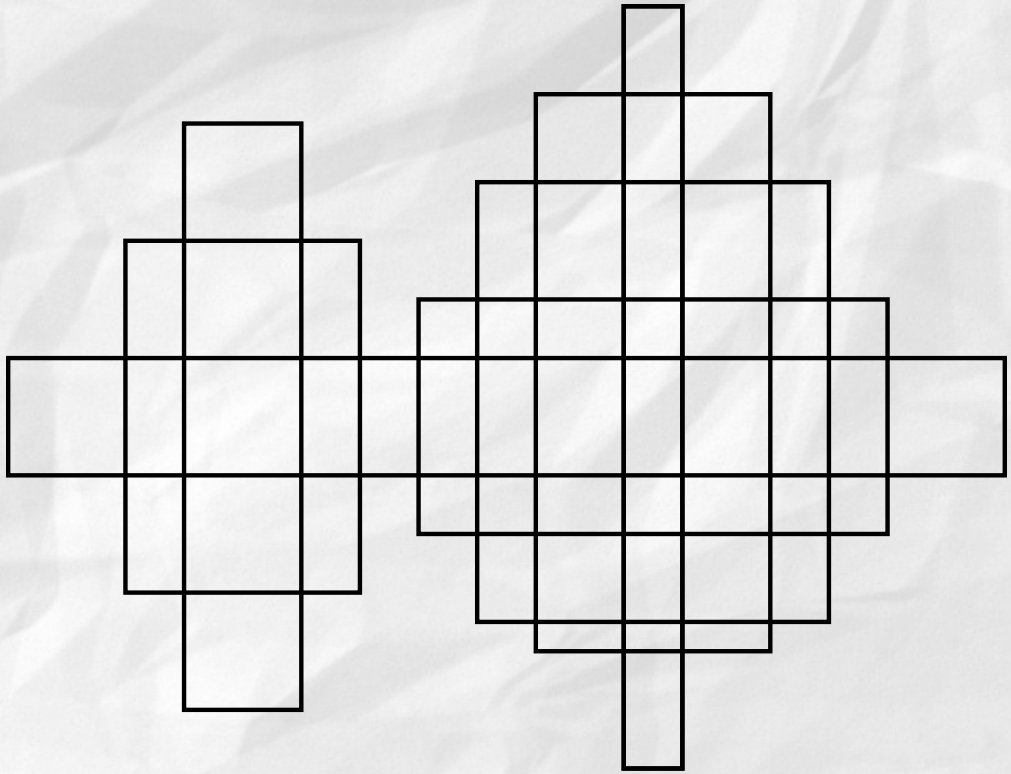
(b) Non-corner intersection



(c) Forget about that! (Why?)

Treating corner and non-corner intersections

If no corner intersections allowed:



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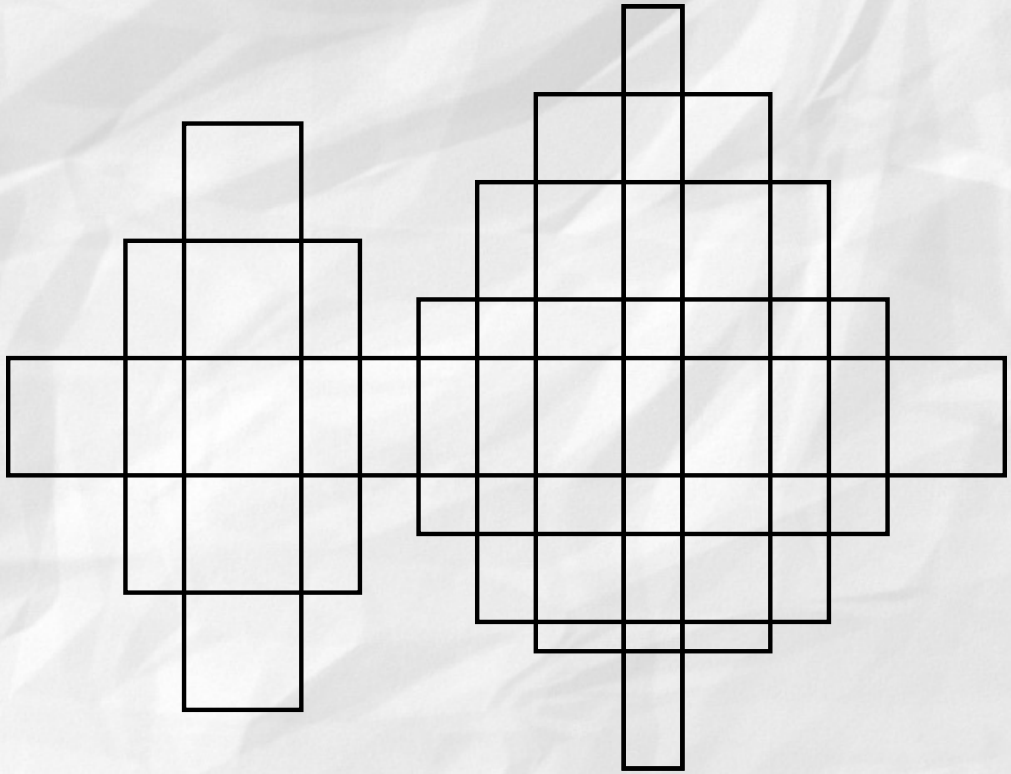

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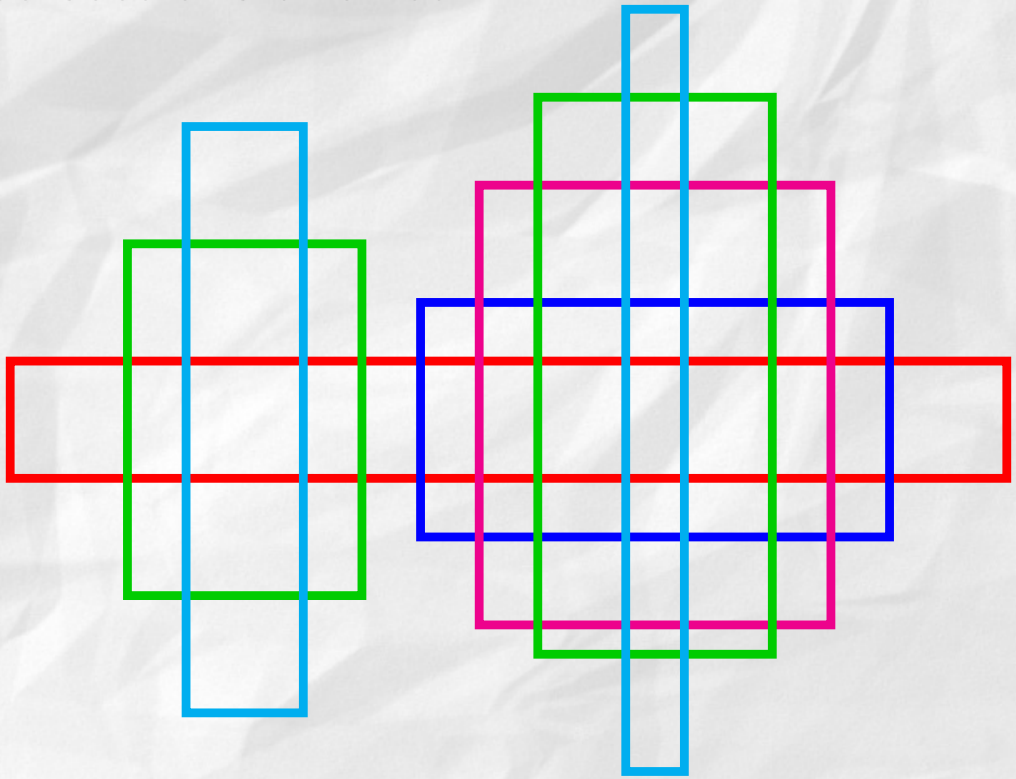

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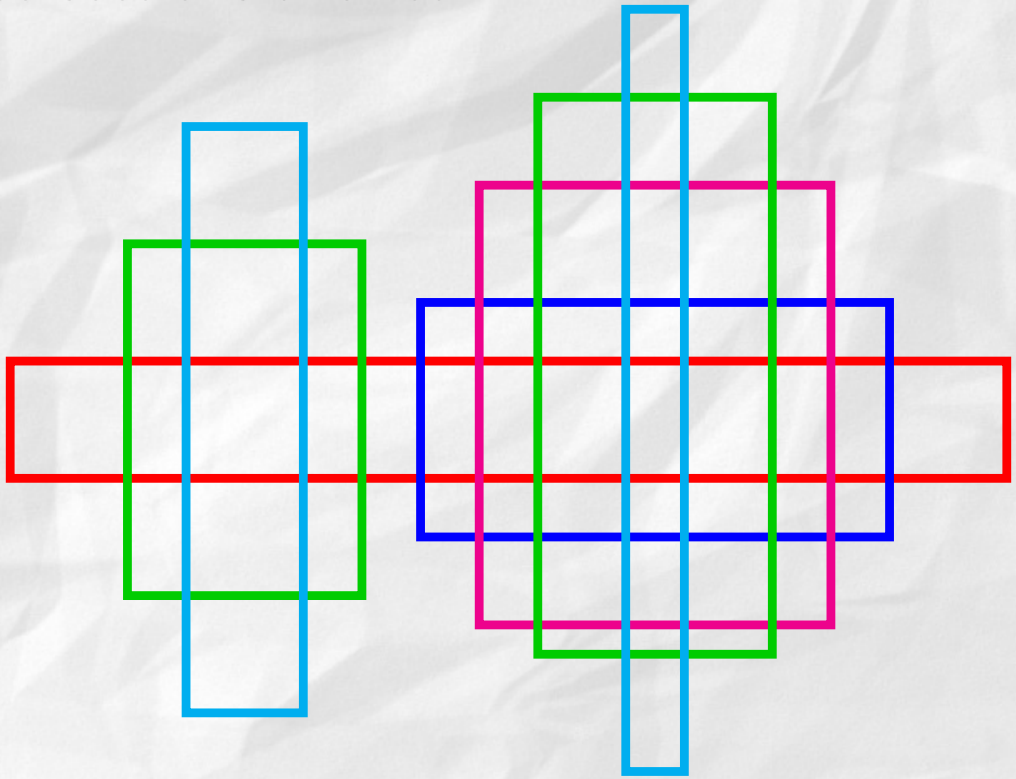

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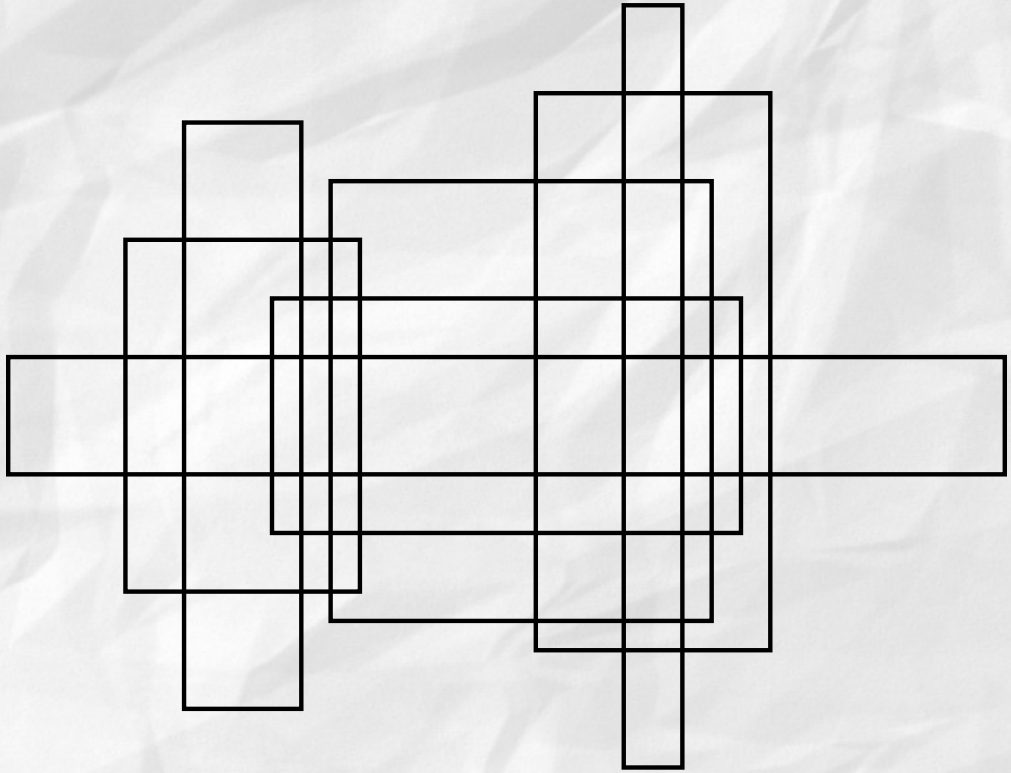
- ◆ Consider the sets $S_j, 0 \leq j < M$: rectangles R which induce j-size cliques in between rectangles R' that have smaller width
- ✔ **Every S_j is an independent set!**

Treating corner and non-corner intersections

What if corner intersections exist?



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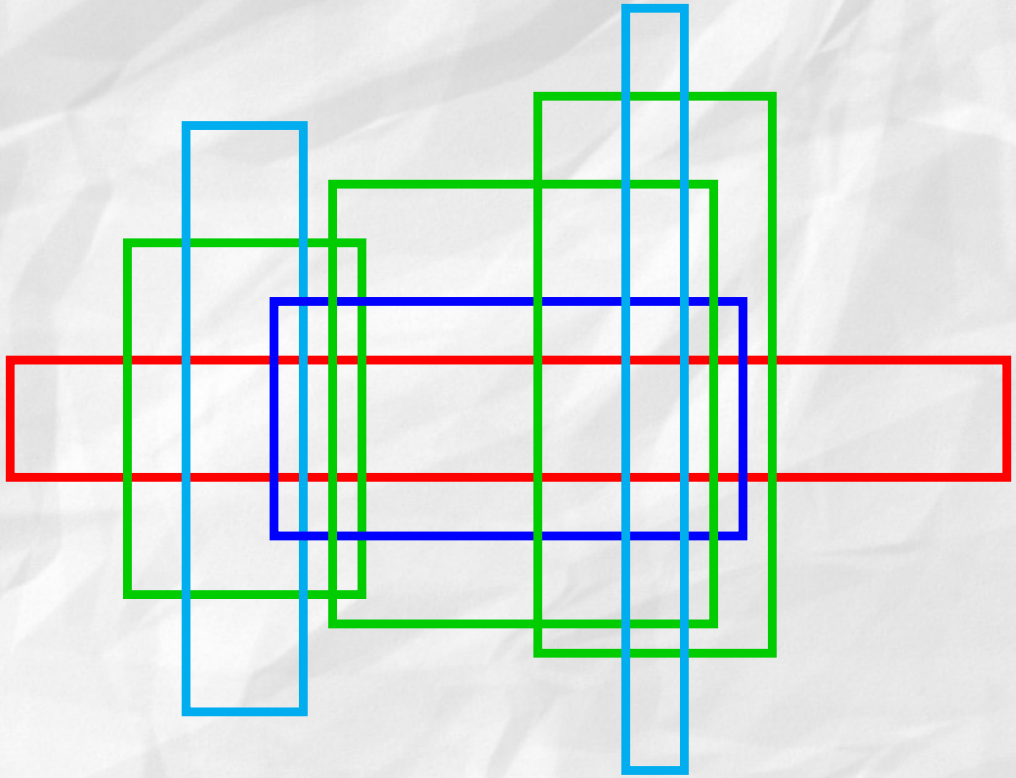

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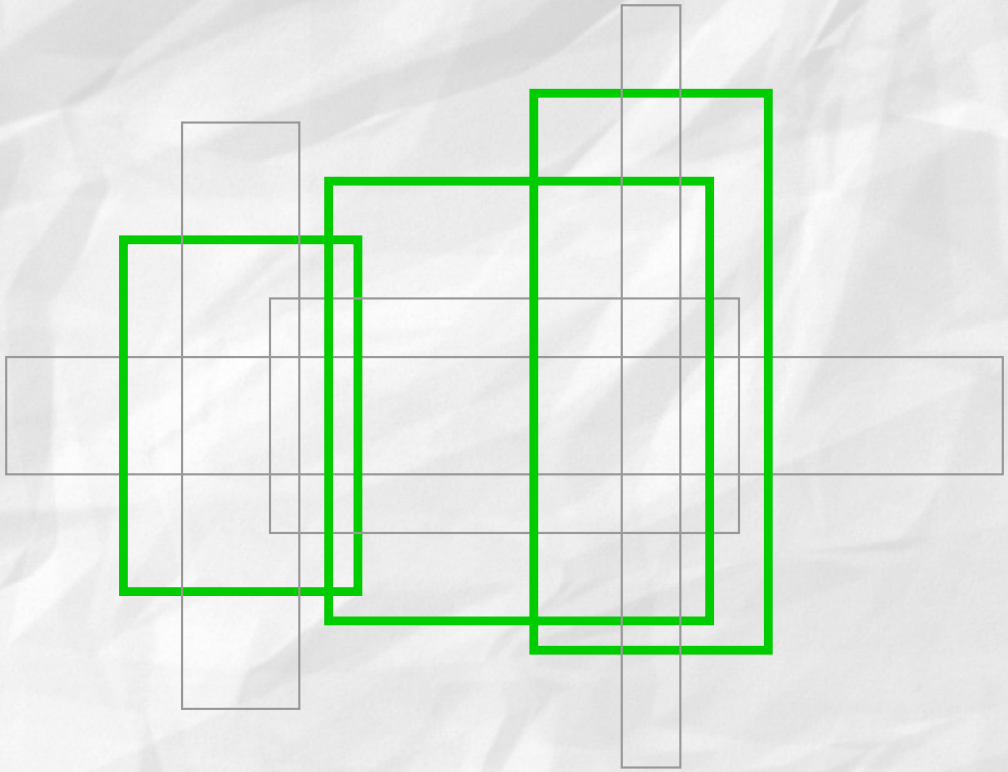

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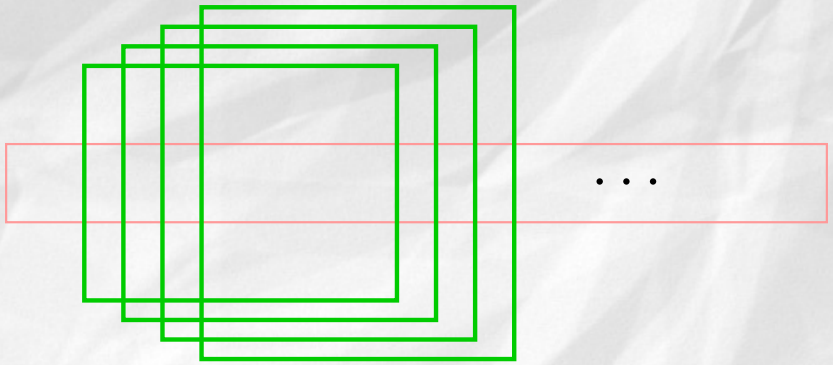
- **No guarantee that S_j is an independent set!**

Treating corner and non-corner intersections

In fact:



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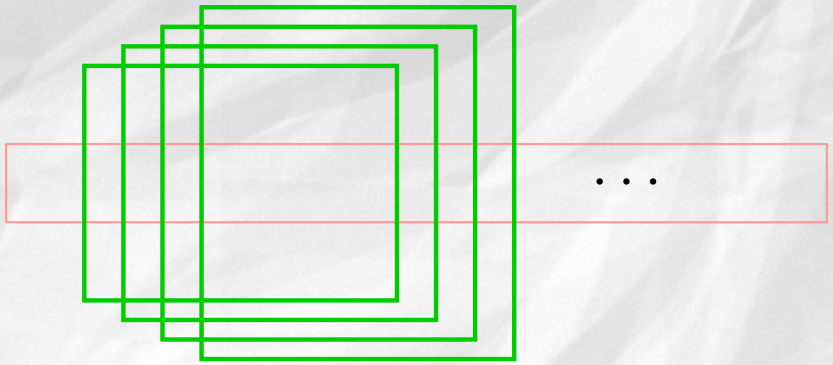

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Nevertheless:

- for all $R \in \mathcal{R}$, define $\boldsymbol{V(R)}$ to include all R' s.t. $R' \cap R$ is a non-corner intersection and $(w_{R'} < w_R)$

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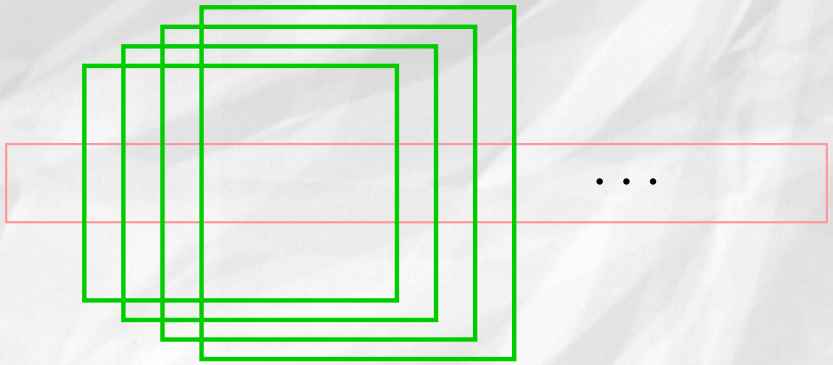

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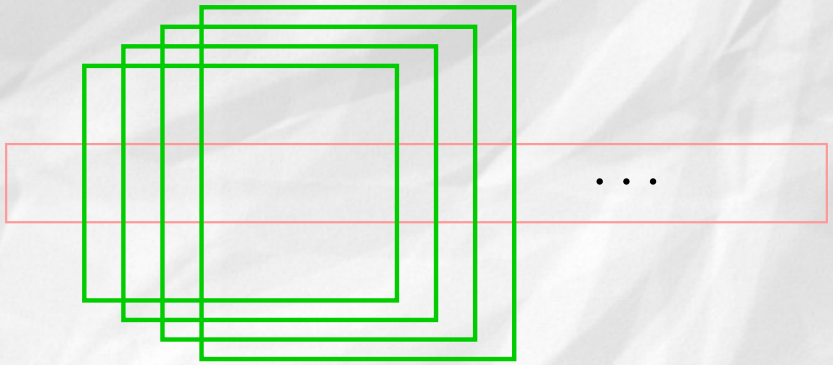

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- and use it to create partitions $\boldsymbol{S_j}$ *according to* $v(R)$

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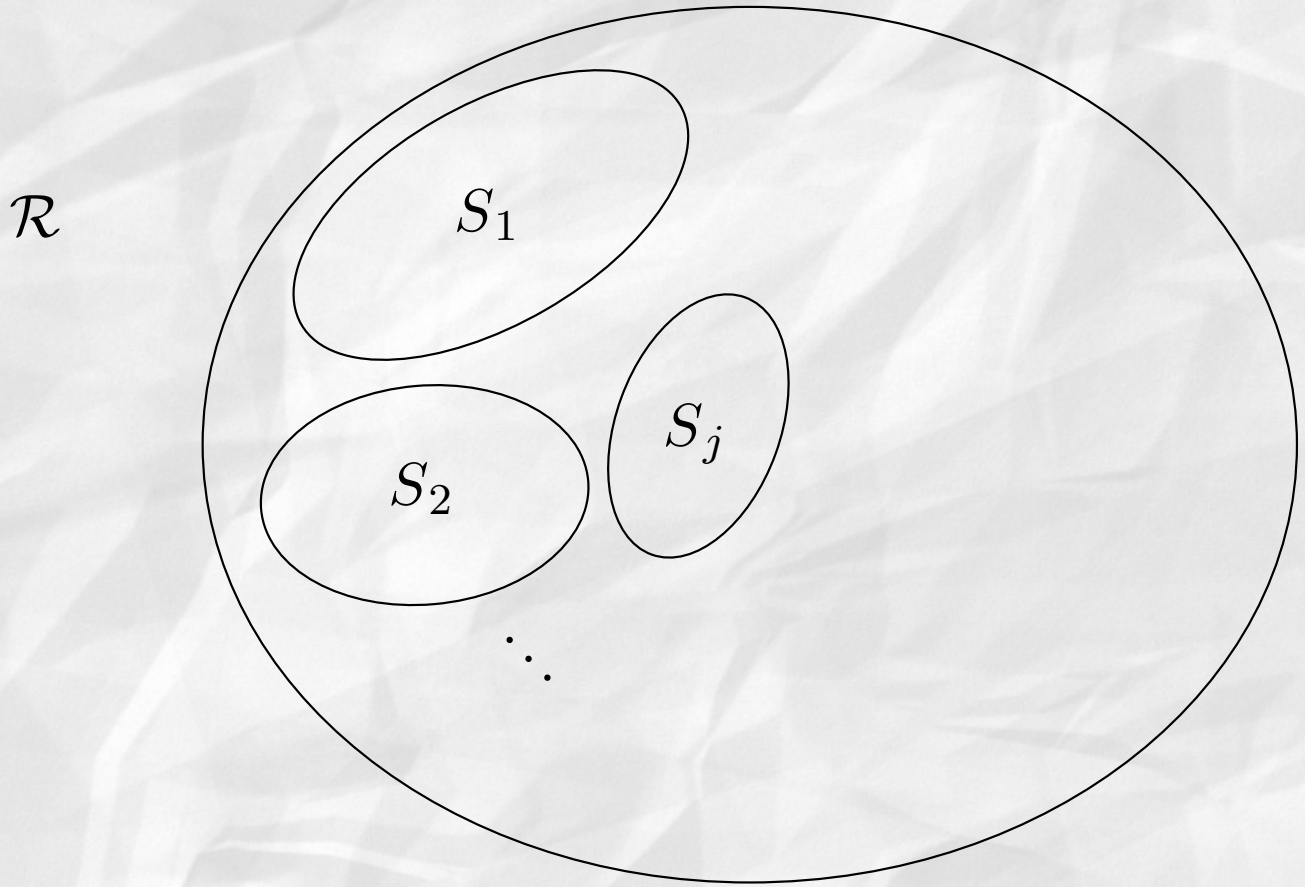
Partition overview



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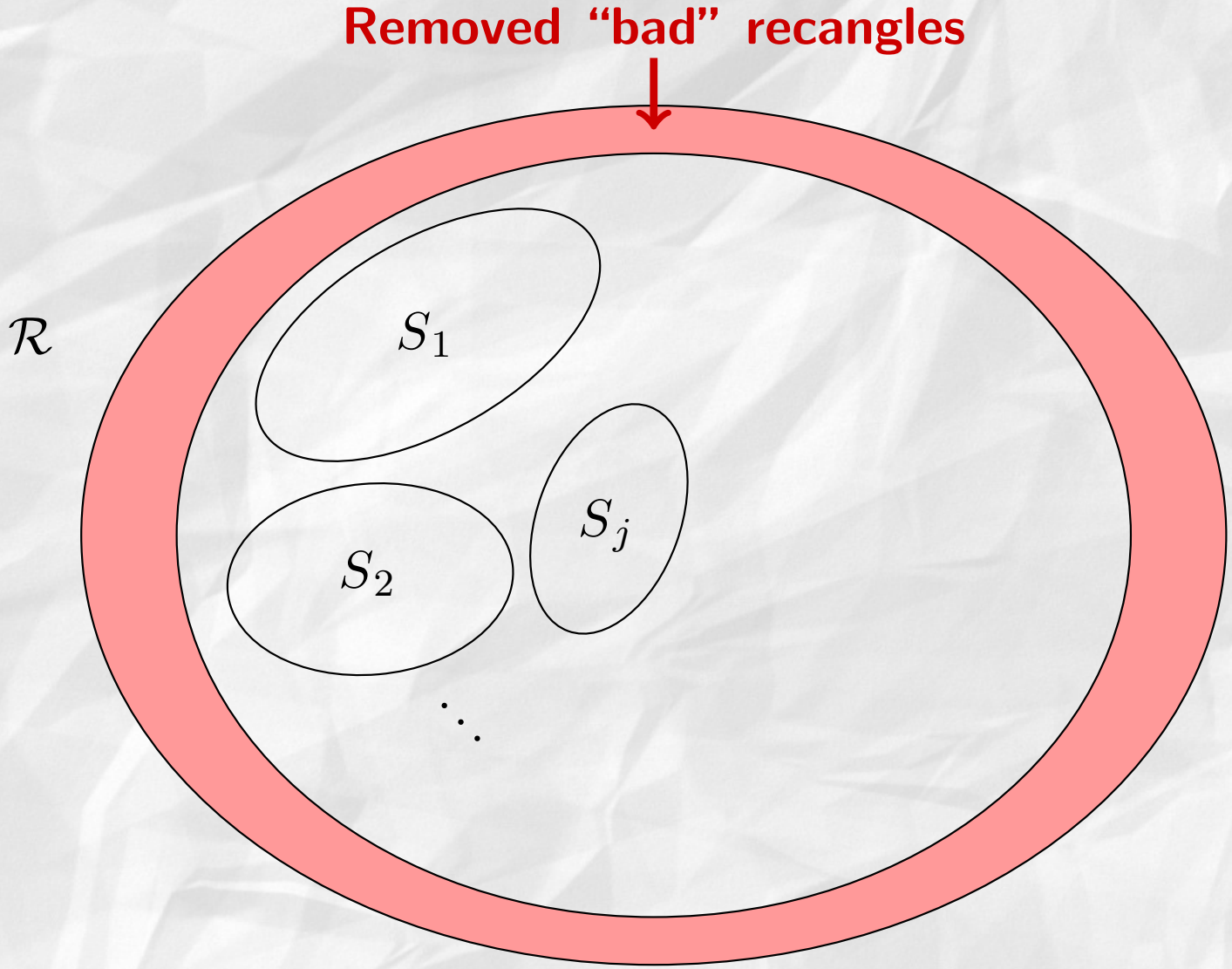

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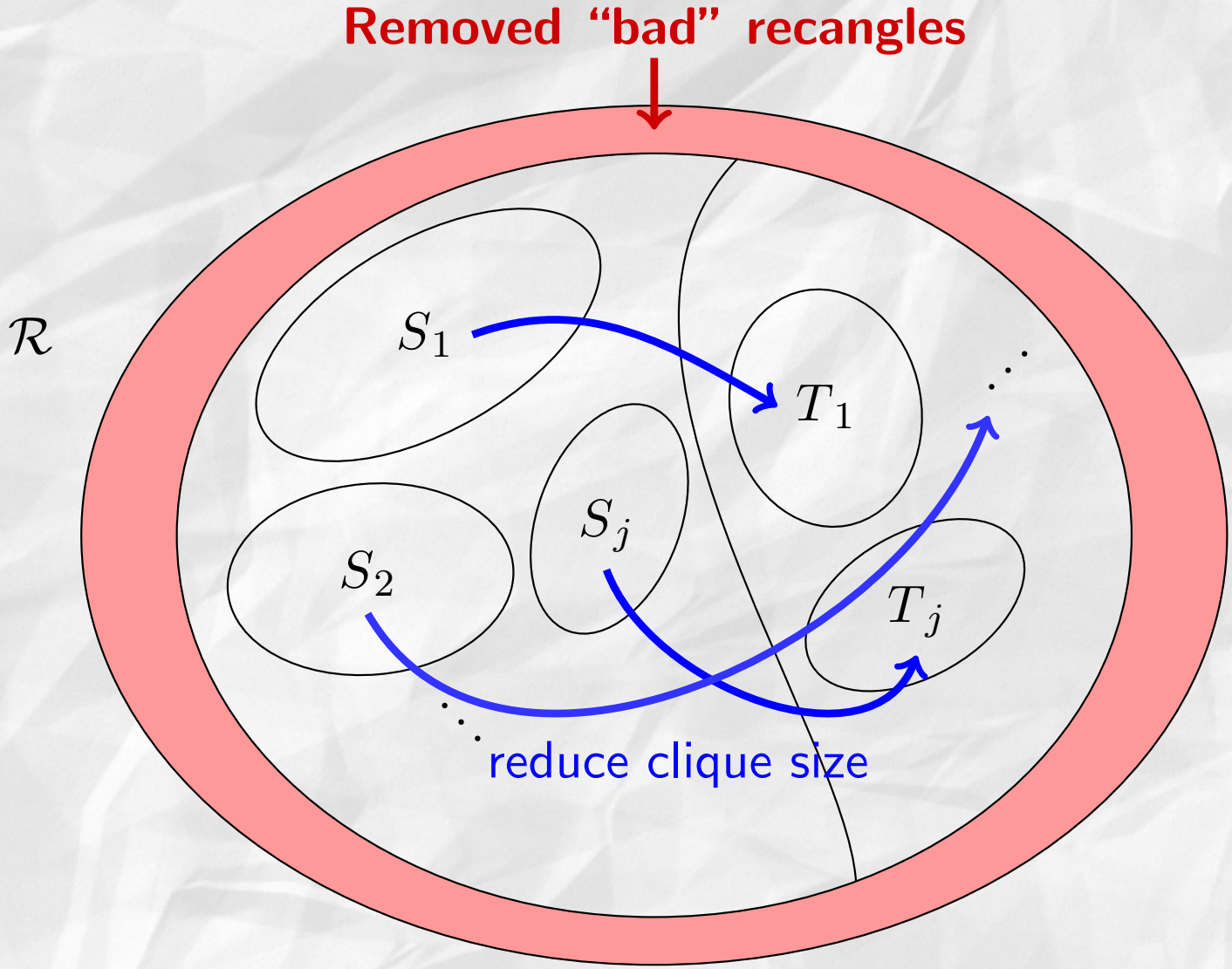

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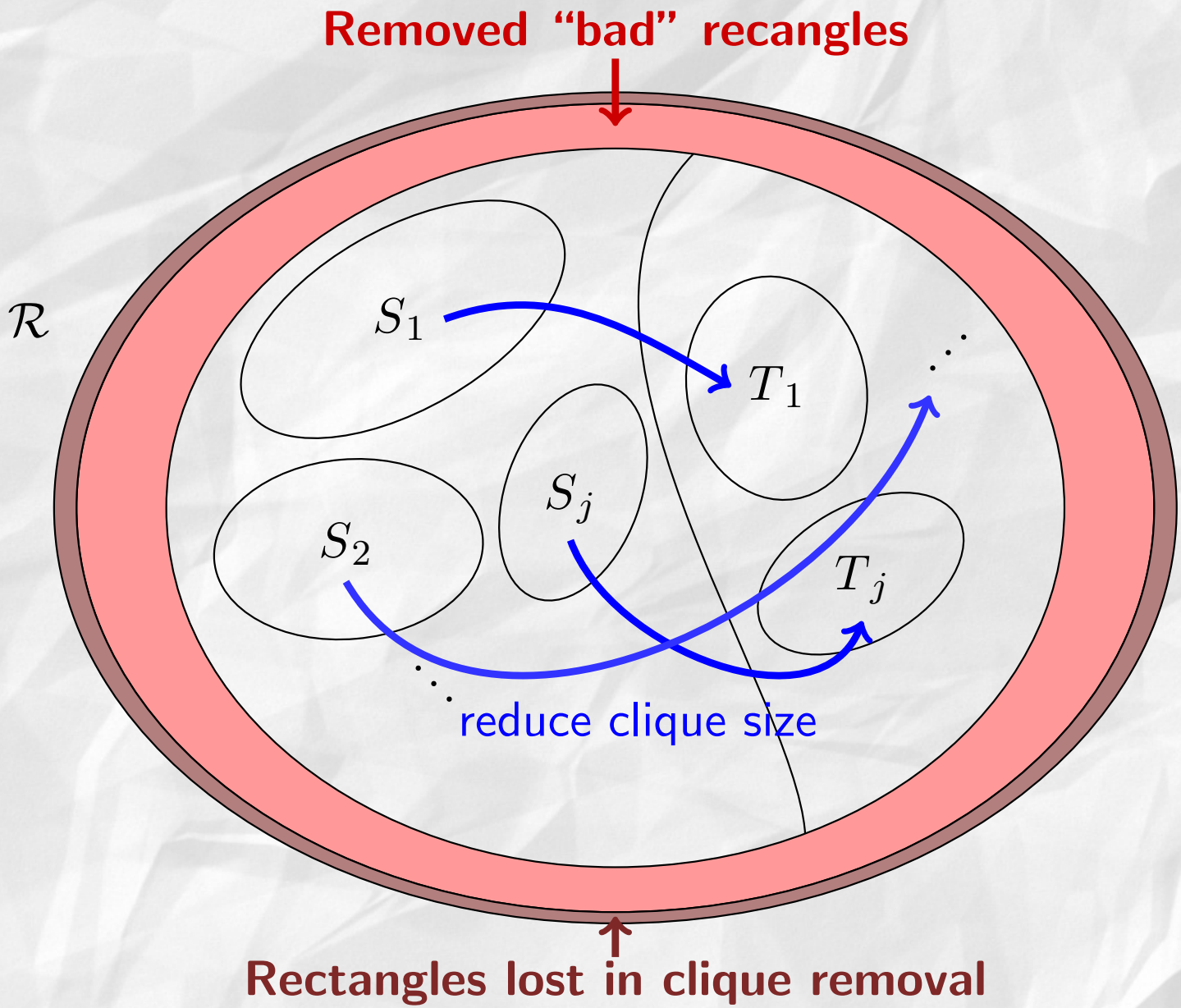

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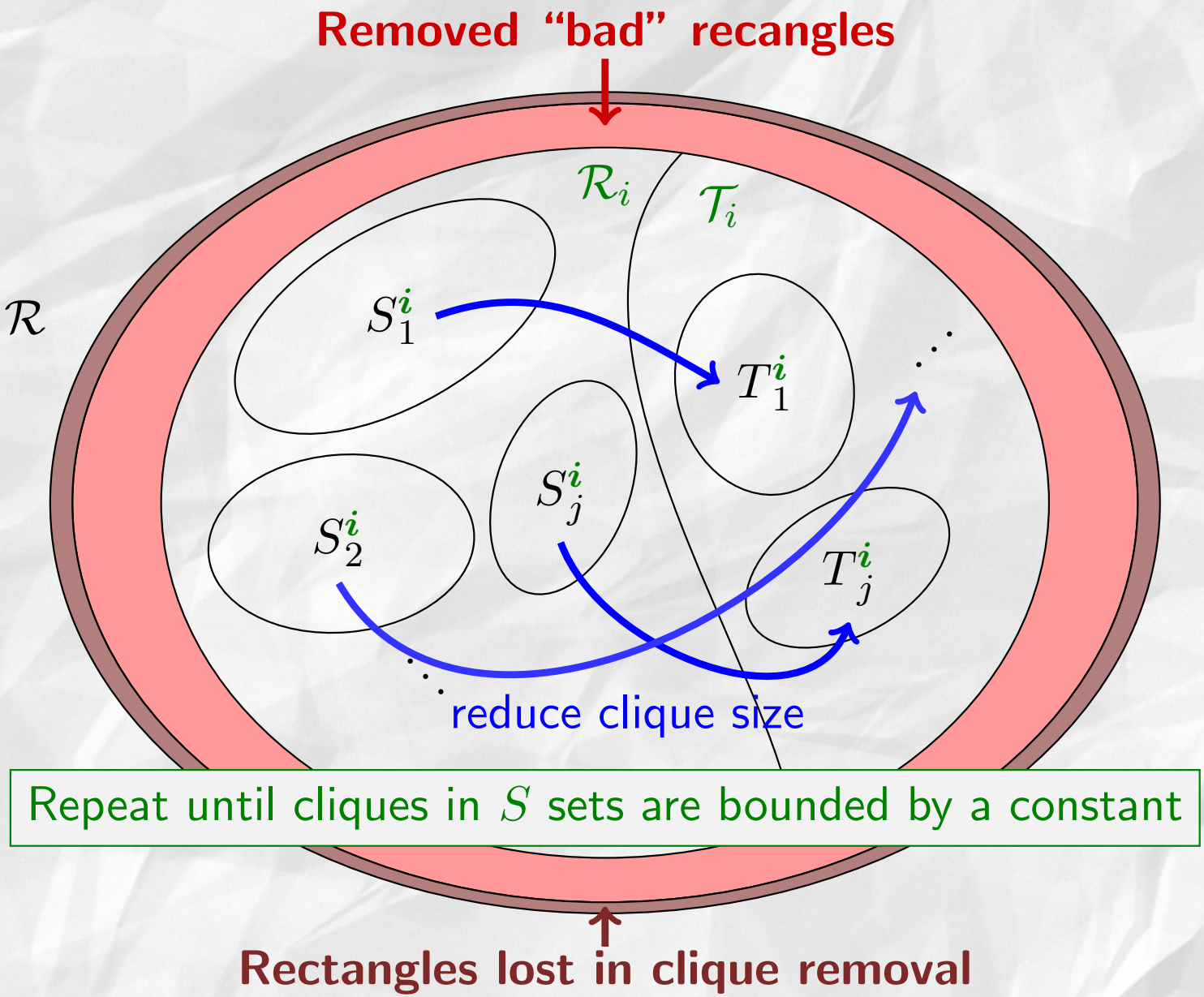

Some more notations

- $\mathcal{R}_i$, $\mathcal{T}_i$, S^i_j, $T^{i \leftarrow \text{iteration}}_{j \leftarrow \text{partition index}}$
- $\bigcup S^i_j = \mathcal{R}_i$, $\bigcup T^i_j = \mathcal{T}_i$
- Every $\mathcal{R}_i$, $\mathcal{T}_i$ is partitioned into $k_i = \beta^i$ subsets ($j \in \{1, ..., k_i\}$)
- $\boldsymbol{W_i}$: total weight of recangles in $\mathcal{R}_i$ **(LP weight, $1/M$ for each)**
- $\boldsymbol{C_i}$: total weight of recangles in $\mathcal{T}_i$ **(charged when moved to some T^i_j)**

Most basic:

$$R \in \boldsymbol{S^i_j} \Leftrightarrow \frac{(j-1)M}{\beta^i} \leq \boldsymbol{v(R)} < \frac{jM}{\beta^i}$$

Algorithm scheme

1. Run LP and create canonical solution $\mathcal{R}$
2. $\mathcal{R}_1 \leftarrow \mathcal{R}$, $\mathcal{T}_1 \leftarrow \emptyset$
3. $i \leftarrow 1$
4. **while** $(\beta^i < M/100)$ **do**
   a. create $\mathcal{R}_{i+1}$, $\mathcal{T}_{i+1}$ so that conditions **C1**–**C4** hold
   b. $i \leftarrow i + 1$
5. find $\max\{\boldsymbol{W_i}, \boldsymbol{C_i}\}$
6. **return** recovered **IS**

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Algorithm's iteration invariants

Take $M = 64 \log n$, $\beta = 20$. Then if the following conditions hold, Proposition 3 is true.

C1. Within T_j^i rectangles do not intersect; an $R \in T_j^i$ does not intersect any $R' \in S_j^i \cup S_{j-1}^i$

C2. Cost of every rectangle in $\mathcal{T}_i$ is $c_R \leq 10^6 \log\log n / \beta^i$

C3. $W_i + C_i \geq OPT(1 - \frac{1}{\log\log n})^i$

C4. The maximum clique of S_j^i is at most $100M/\beta^i$

Proposition 3.

There is an $h = \Theta(\log\log n)$ *such that the output* $\mathcal{R}_{h+1}, \mathcal{T}_{h+1}$ *quarantees the* $\log\log n$ *approximation for* `MISR`.

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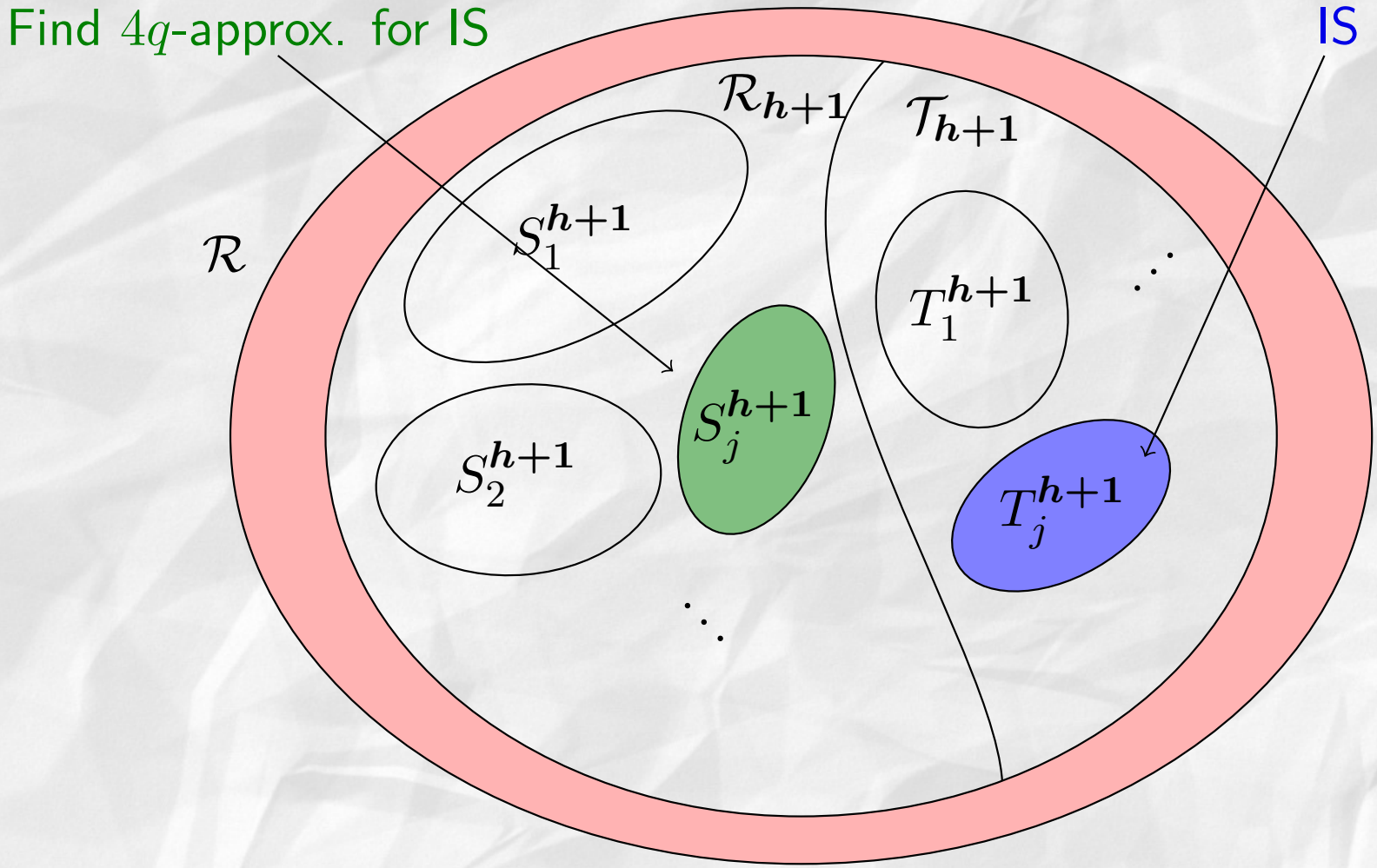

Algorithm's output is the desired

- **Stop when $\beta^h \geq M/100 \Rightarrow h+1 = \Theta(\log\log n)$**
- Take **C3**: $W_{h+1} + C_{h+1} \geq OPT(1 - \frac{1}{\log\log n})^{\Theta(\log\log n)} \geq \frac{OPT}{e}$
- That means $W_{h+1} + C_{h+1} = \boldsymbol{\Omega(OPT)}$, or, in other words, **not both W_{h+1} and C_{h+1} are asymptotically smaller than OPT**
- Case $W_{h+1} = \Omega(OPT)$:
  - $|\mathcal{R}_{h+1}| = \Omega(M \cdot OPT)$ (uniform rectangles' weight $1/M$)
  - pigeonhole principle: **there is a set** S_j^{h+1} with $\Omega(\frac{M \cdot OPT}{M/100}) = \boldsymbol{\Omega(OPT)}$ rectangles and
    - **bounded size clique** (**C4**)
    - apply Theorem 1 ($4q$-approx. where q is the largest clique)

...This is the good case, giving a constant factor approximation...

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  - $|\mathcal{R}_{h+1}| = \boldsymbol{\Omega(M \cdot OPT)}$ (uniform rectangles' weight $1/M$)
  - pigeonhole principle: **there is a set** S_j^{h+1} with $\Omega(\frac{M \cdot OPT}{M/100}) = \boldsymbol{\Omega(OPT)}$ rectangles and
  - **bounded size clique** (**C4**)
  - apply Theorem 1 ($4q$-approx. where q is the largest clique)

...This is the good case, giving a constant factor approximation...

Algorithm's output is the desired

- **Stop when** $\beta^h \geq M/100 \Rightarrow h+1 = \Theta(\log\log n)$
- Take **C3**: $W_{h+1} + C_{h+1} \geq OPT(1 - \frac{1}{\log\log n})^{\Theta(\log\log n)} \geq \frac{OPT}{e}$
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- **Case $C_{h+1} = \Omega(OPT)$:**
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Algorithm scheme

1. Run LP and create canonical solution $\mathcal{R}$
2. $\mathcal{R}_1 \leftarrow \mathcal{R}$, $\mathcal{T}_1 \leftarrow \emptyset$
3. $i \leftarrow 1$
4. **while** ($\beta^i < M/100$) **do**
   a. **create** $\mathcal{R}_{i+1}$, $\mathcal{T}_{i+1}$ **so that conditions C1–C4 hold**
   b. $i \leftarrow i + 1$
5. find $\max\{\boldsymbol{W_i}, \boldsymbol{C_i}\}$
6. **return** recovered **IS**

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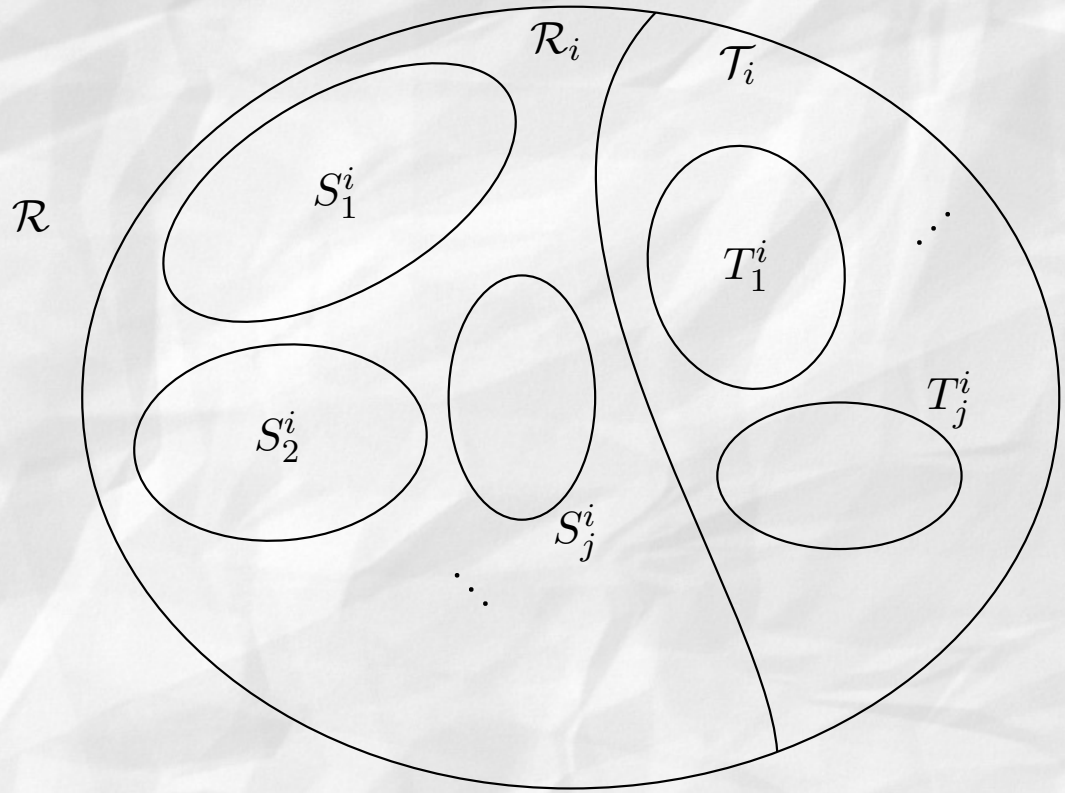
Refine partitions



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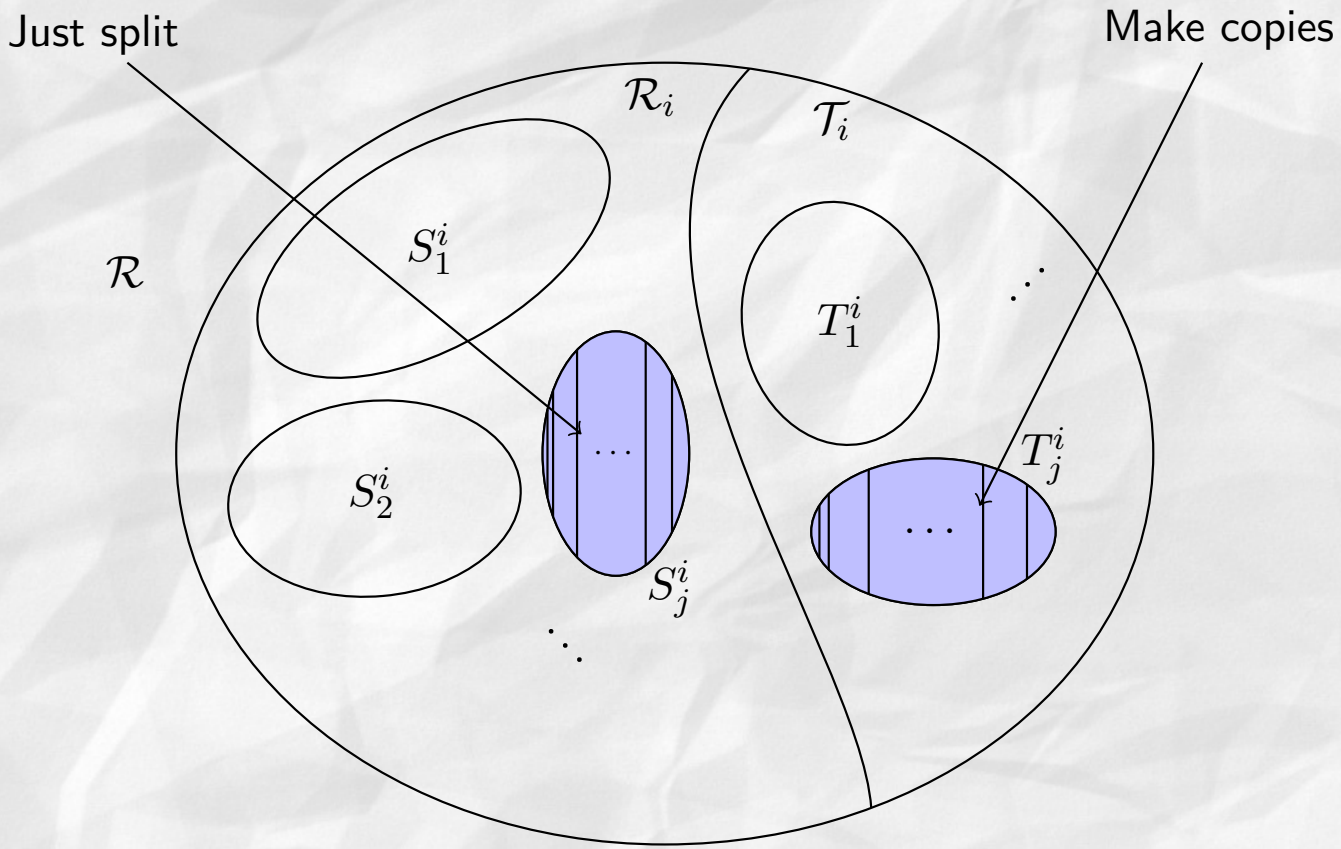

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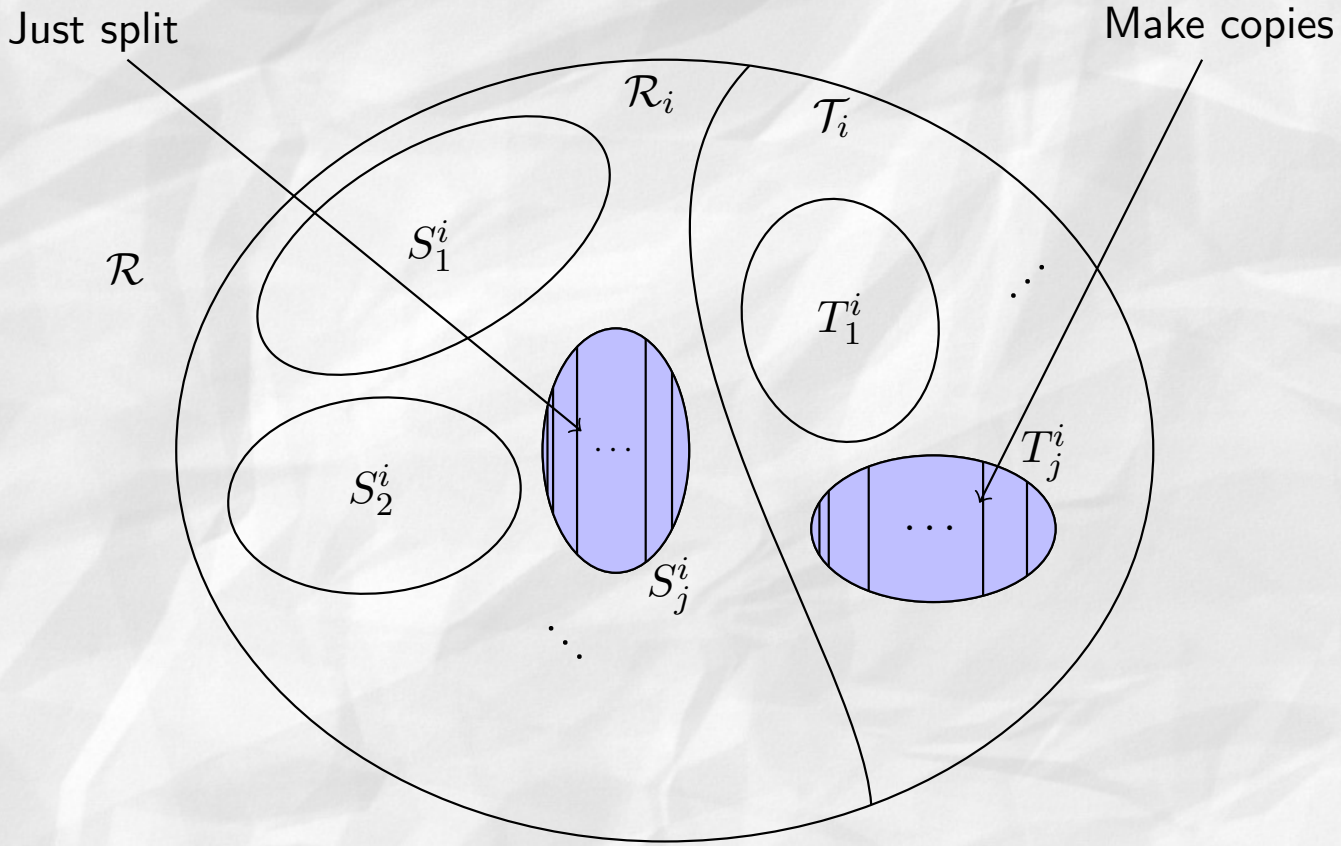

✔Easy to see that **C1**–**C3** hold.

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Bad rectangles

Getting closer to the core of the work, we need the following:

Claim 4 (Deriving from [8]).

For any set of rectangles $\mathcal{H}$ with max. clique size q, the number of **(γ, q)-bad** *rectangles, i.e.* **those which have corner intersections with at least γq recangles in $\mathcal{H}$,** *is at most* $\frac{4}{\gamma}|\mathcal{H}|$.

- It suffices to consider parameters $\gamma = 12 \log\log n$ and $q = 300M/\beta^i$ for rectangles in S_j^{i+1} and set set $\mathcal{H}$ to be $S_j^{i+1} \cup S_{j+1}^{i+1} \cup S_{j+2}^{i+1}$.
- We then remove at most $\frac{|S_j^{i+1} \cup S_{j+1}^{i+1} \cup S_{j+2}^{i+1}|}{3 \log\log n}$ from each S_j^{i+1} (Claim 4).
- Summing for all j, the numerator becomes at most $3|\mathcal{R}_{i+1}|$;
- **C1**,**C2** are not affected while **C3 still holds**: C_{i+1} is unchanged, W_{i+1} was decreased by a factor of $\log\log n$ at most.

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α-cover

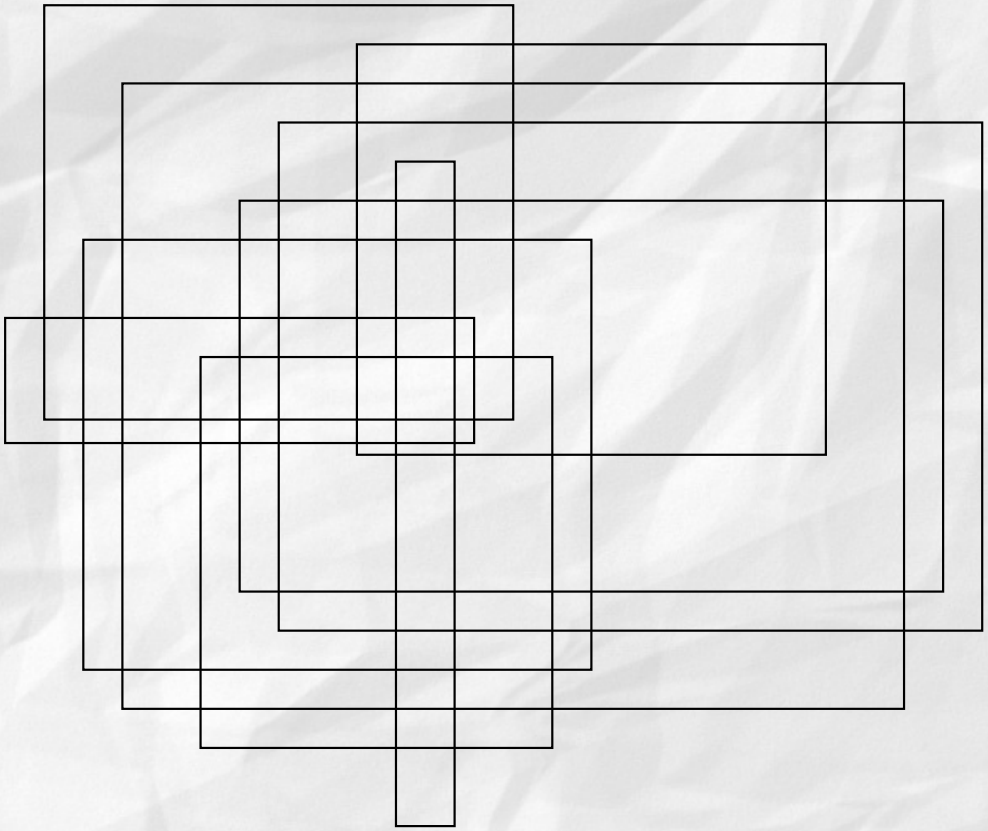
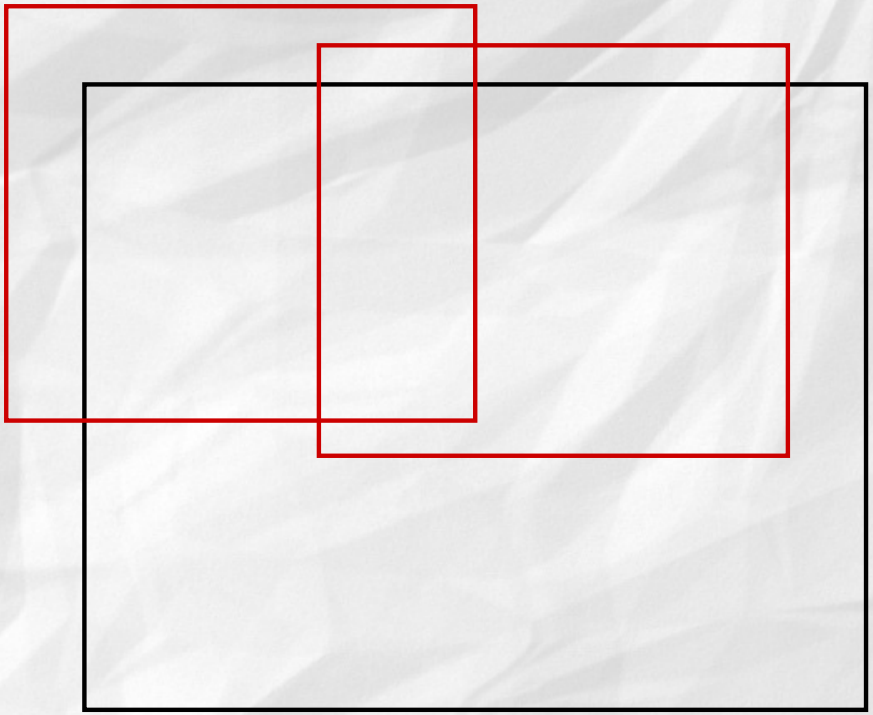


Figure:

α-cover



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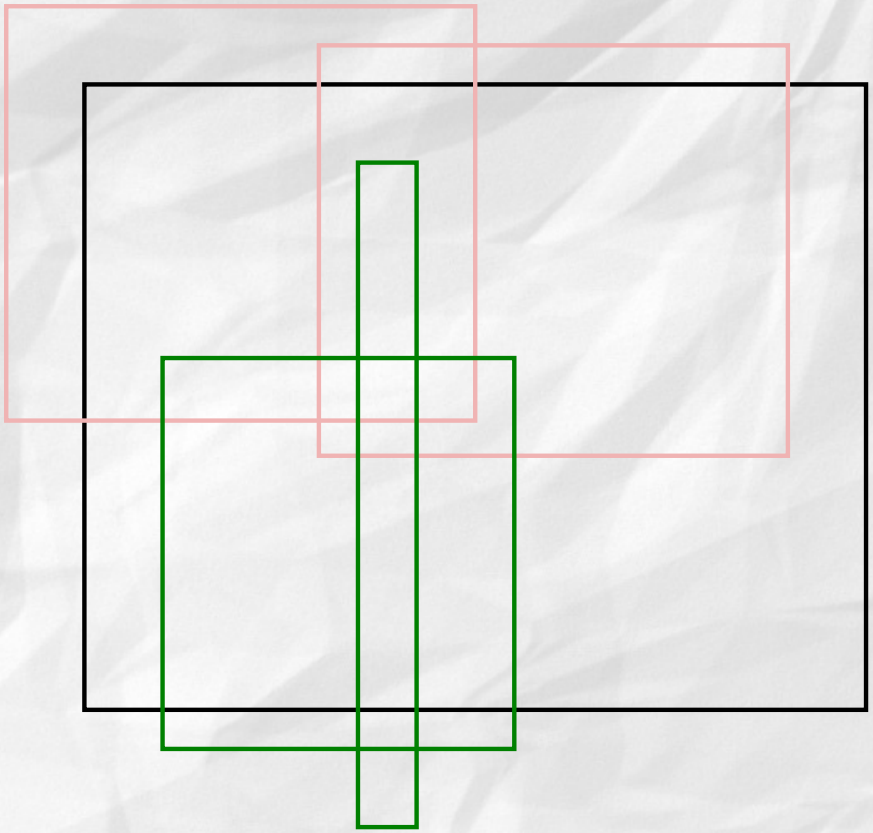

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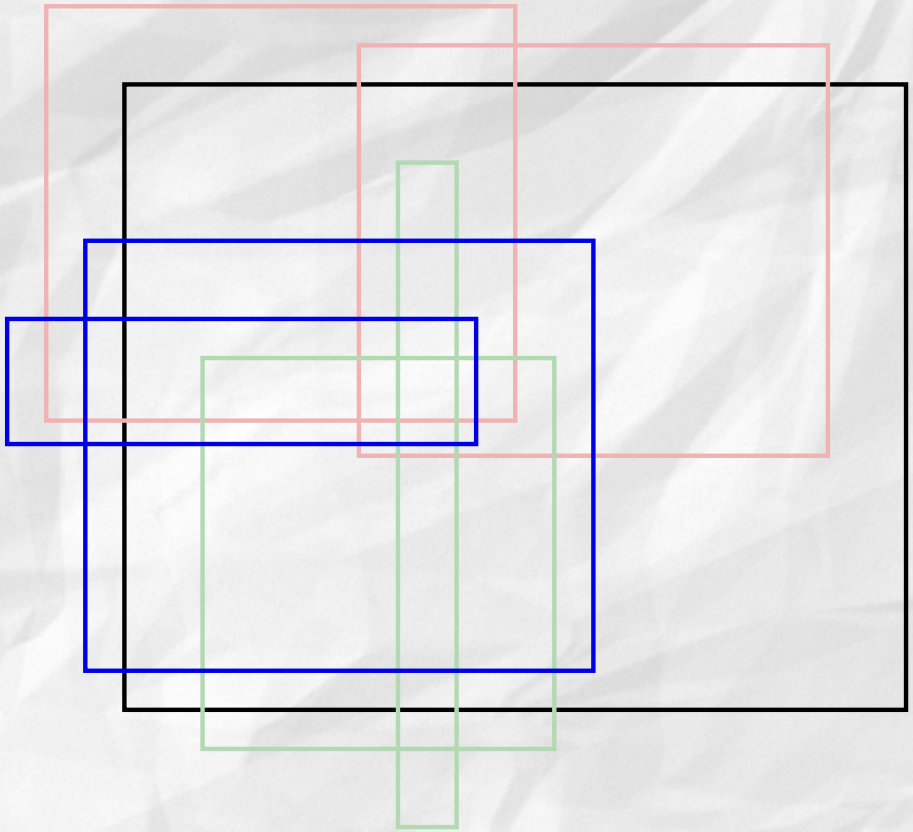

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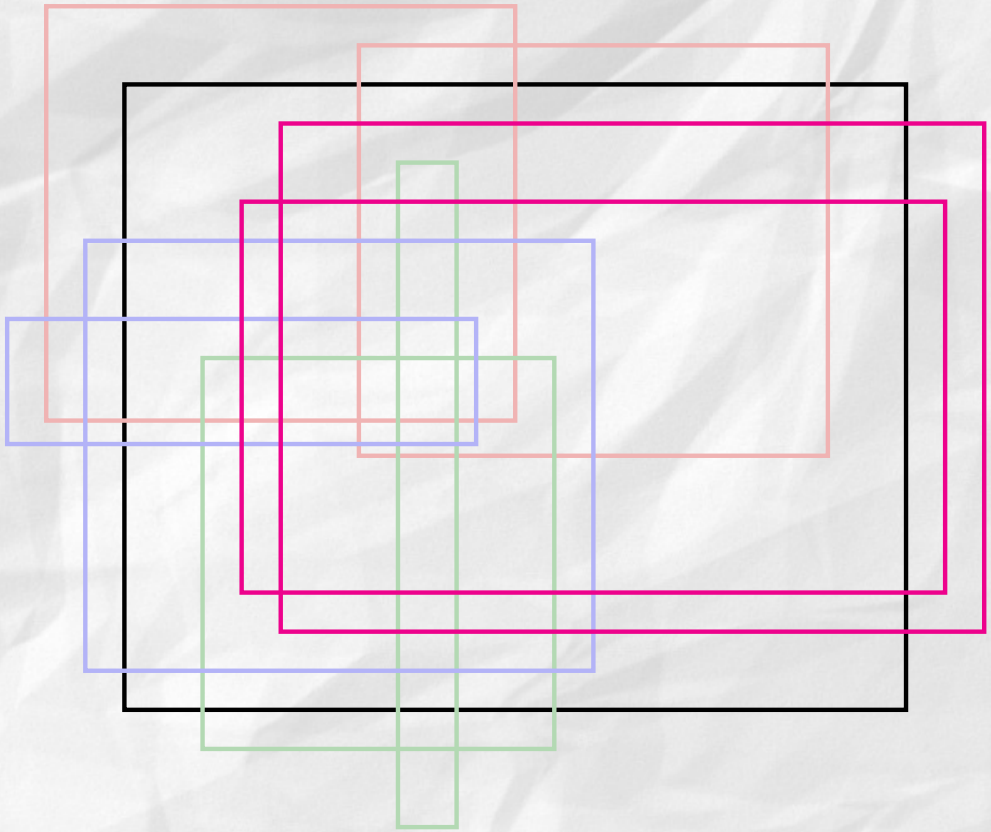

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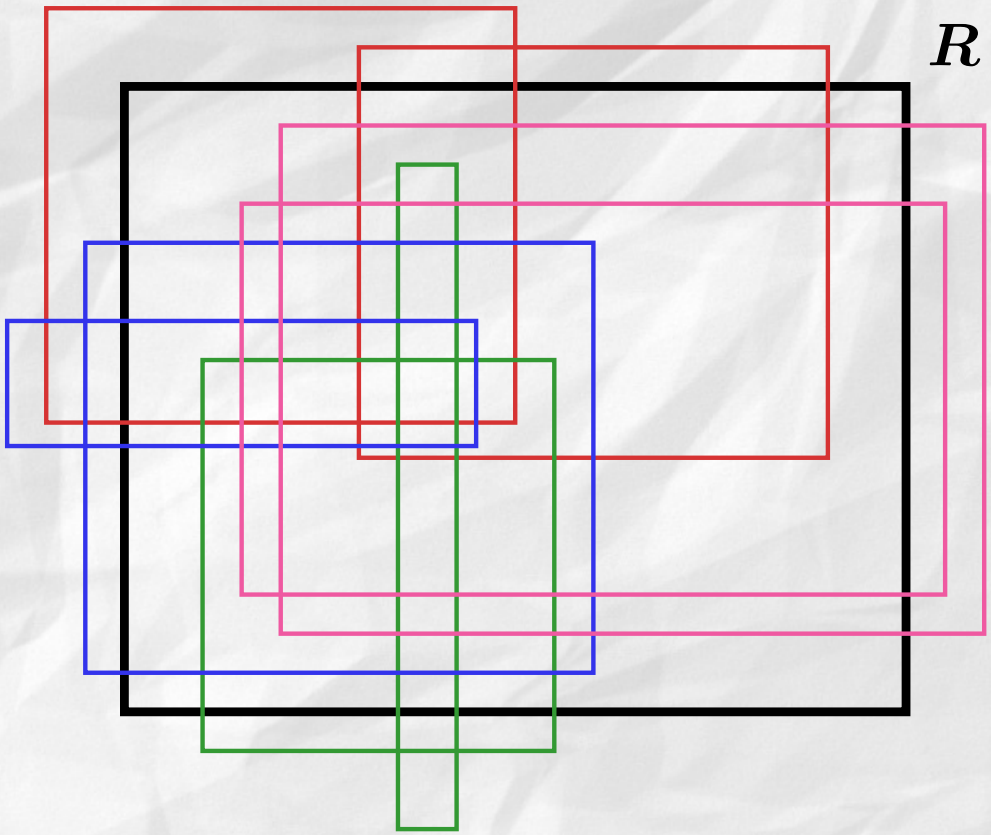

Figure: α-cover, $\alpha(R) = 2$

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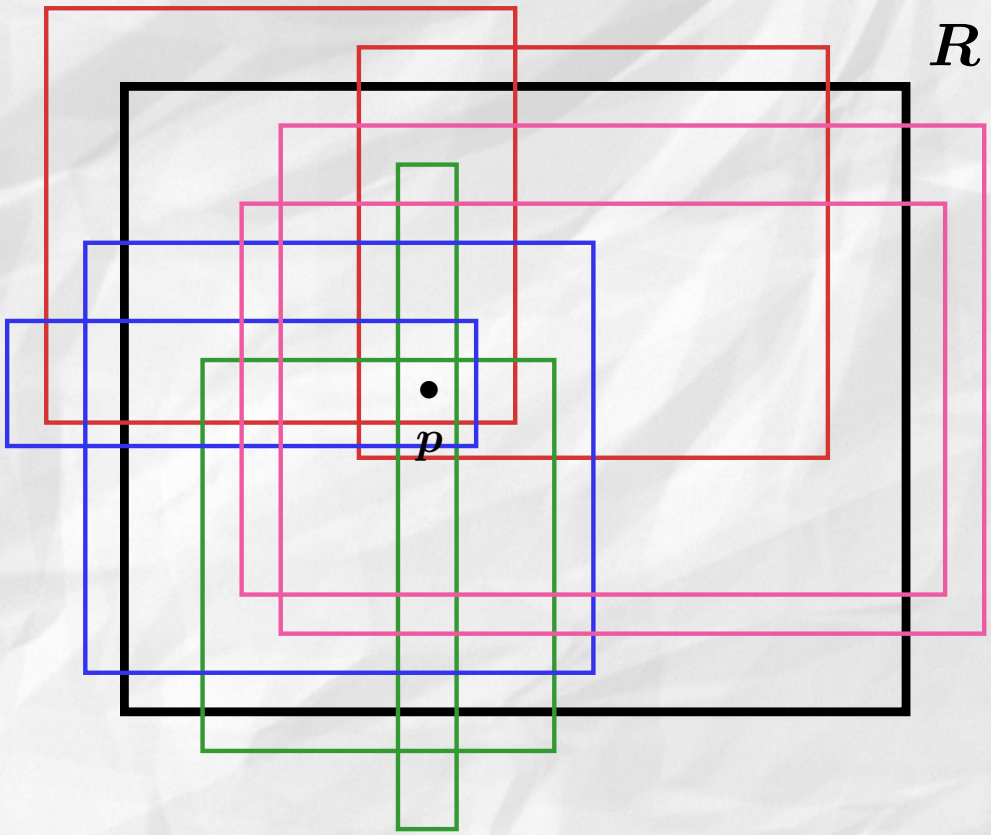

Figure: α-cover, $\alpha(R) = 2$

Claim 5.

If S_j contains a clique of size c then $\exists R \in S_j$ s.t. $\alpha(R) \geq \lfloor \frac{c}{4} \rfloor + 1$

Round(R) for removing the cliques

1. **Move** $\boldsymbol{R}$ to T_{j+2}
2. Remove all rectanges in $S_j \cup S_{j+1} \cup S_{j+2}$ containing $\boldsymbol{p}$
3. Remove all rectanges in $S_j \cup S_{j+1} \cup S_{j+2}$ having **corner intersections** with at least one of $\{R, R_l^1, R_r^1, R_l^2, R_r^2\}$
4. **Charge** R in T_{j+2} with the sum of the weights removed.

Observe:

- C3 **and C4** now hold!

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C2. Cost of every rectangle in $\mathcal{T}_i$ is $\boldsymbol{c_R \leq 10^6 \log\log n/\beta^i}$

- For C2, remember we have removed our bad rectangles, therefore the rectangles removed in Step 3 of *Round(R)* now intersect with at most $\gamma q = \Theta(\log\log n) \cdot q$ rectangles.

  Rectangles removed in Step 2 of *Round(R)* where contained in bounded-size clique.

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Checking condition C1

C1. Within T_j^i rectangles do not intersect; an $R \in T_j^i$ does not intersect any $R' \in S_j^i \cup S_{j-1}^i$

- Before *Round(R)*, **C1** holds. It is enough to prove:

Lemma 6.

Whenever rectangle $R \in S_j$ is added to T_{j+2} due to Round(R), it does not intersect any rectangle in $T_{j+2} \cup S_{j+2} \cup S_{j+1}$. **Moreover, there is no rectangle $R' \in S_j$ that intersects R in a non-corner manner, and whose width is greater than the width of R.**

Claim 7.

Let R' be any rectangle intersecting R in a non-corner manner, such that the width of R' is greater than that of R. Then $v(R') \geq v(R) + \alpha(R)/4 \geq v(R) + 5M/k$. It follows that $R' \notin S_j \cup S_{j+1} \cup S_{j+2}$.

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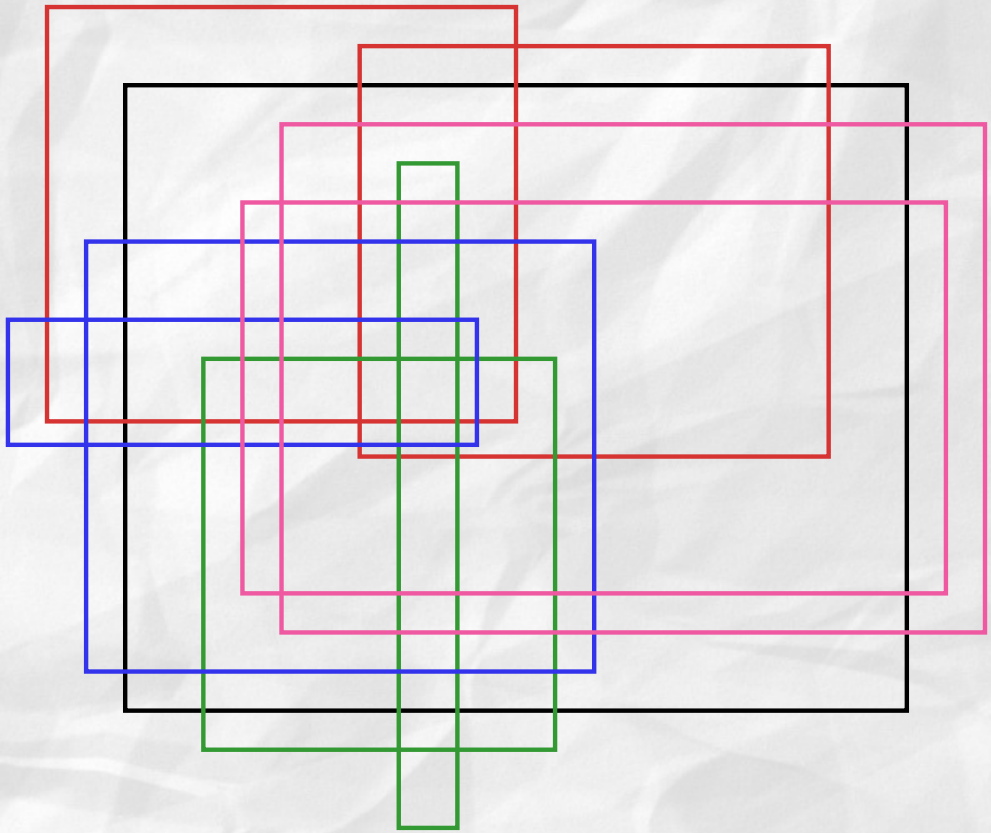

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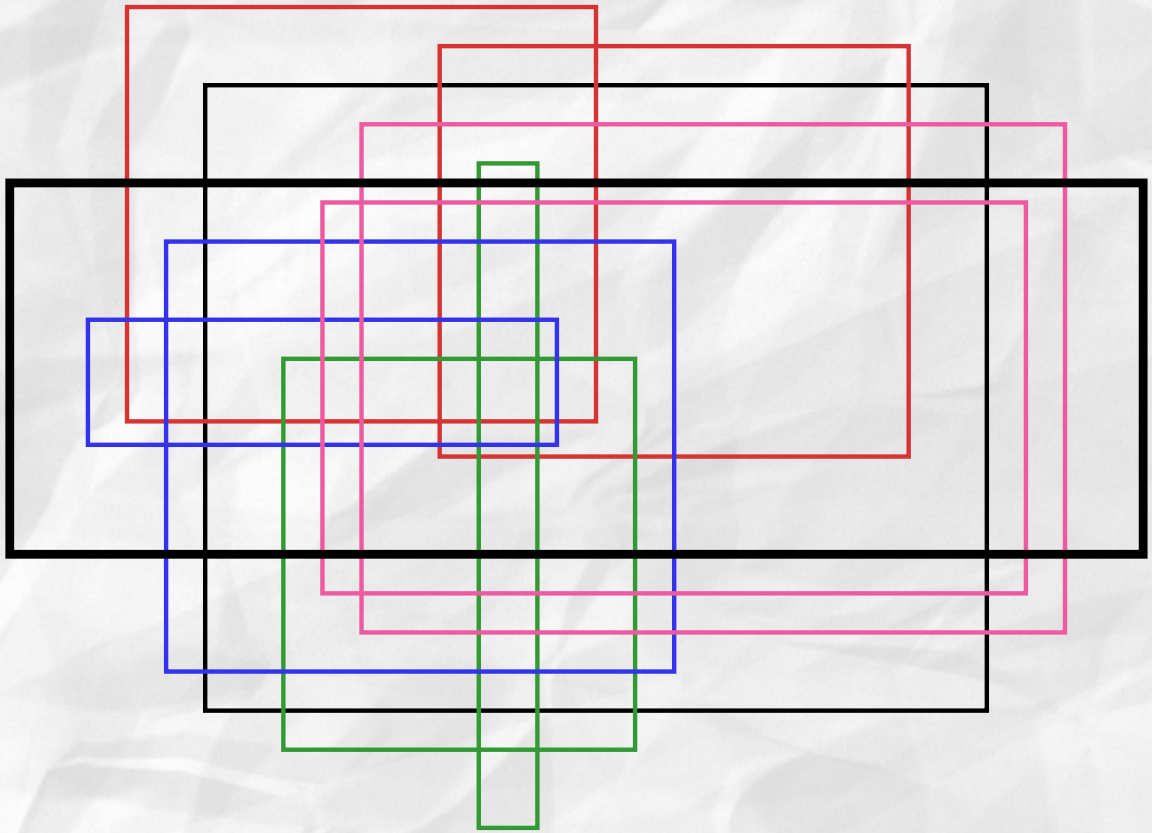

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Generalization in higher dimensions

Theorem 8 (Chalermsook and Chuzhoy (2009)).

If there is a factor $f(n)$-approximation algorithm for $\texttt{MISB}_d$, then there is an $O(f(n)\log n)$-approximation algorithm for $\texttt{MISB}_{d+1}$.

It follows that there is an $O((\log n)^{d-2}\log\log n)$-approximation algorithm for MISB *on d dimensions.*

This improves the $O((\log n)^{d-1})$-approximation given by Berman et al., [2].

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- Approximation status remains open (**PTAS?**)
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The end

Thank you!

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