Efficient Online Learning of Optimal Rankings: Dimensionality Reduction via Gradient Descent

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Abstract

We consider a natural model of online preference aggregation, where sets of preferred items R_1, R_2, \ldots, R_t along with a demand for k_t items in each R_t , appear online. Without prior knowledge of (R_t, k_t) , the learner maintains a ranking π_t aiming that at least k_t items from R_t appear high in π_t . This is a fundamental problem in preference aggregation with applications to, e.g., ordering product or news items in web pages based on user scrolling and click patterns. The widely studied *Generalized Min-Sum-Set-Cover* (GMSSC) problem serves as a formal model for the setting above. GMSSC is NP-hard and the standard application of no-regret online learning algorithms is computationally inefficient, because they operate in the space of rankings. In this work, we show how to achieve low regret for GMSSC in polynomial-time. We employ dimensionality reduction from rankings to the space of doubly stochastic matrices, where we apply Online Gradient Descent. A key step is to show how subgradients can be computed efficiently, by solving the dual of a configuration LP. Using oblivious deterministic and randomized rounding schemes, we map doubly stochastic matrices back to rankings with a small loss in the GMSSC objective.

1 Introduction

In applications where items are presented to the users sequentially (e.g., web search, news, online shopping, paper bidding), the item ranking is of paramount importance (see e.g., [38, 12, 14, 43, 7]). More often than not, only the items at the first few slots are immediately visible and the users may need to scroll down, in an attempt to discover items that fit their interests best. If this does not happen soon enough, the users get disappointed and either leave the service (in case of news or online shopping, see e.g., [8]).

To mitigate such situations and increase user retention, modern online services highly optimize item rankings based on user scrolling and click patterns. Each user t is typically represented by her set of preferred items (or item categories) R_t . The goal is to maintain an item ranking π_t online such that each new user t finds enough of her favorite items at relatively high positions in π_t ("enough" is typically user and application dependent). A typical (but somewhat simplifying) assumption is that the user dis-utility is proportional to how deep in π_t the user should reach before that happens.

The widely studied *Generalized Min-Sum Set Cover* (GMSSC) problem (see e.g., [28] for a short survey) provides an elegant formal model for the practical setting above. In (the offline version of) GMSSC, we are given a set $U = \{1, \ldots, n\}$ of n items and a sequence of requests $R_1, \ldots, R_T \subseteq U$. Each request $R \subseteq U$ is associated with a demand (or covering requirement) $K(R) \in \{1, \ldots, |R|\}$. The access cost of a request R wrt. an item ranking (or permutation) π is the index of the K(R)-th element from R in π . Formally,

 $\operatorname{AccessCost}(\pi, R) = \{ \text{the first index up to which } K(R) \text{ elements of } R \text{ appear in } \pi \}.$ (1)

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The goal is to compute a permutation $\pi^* \in [n!]$ of the items in U with minimum total access cost, i.e., $\pi^* = \arg \min_{\pi \in [n!]} \sum_{t=1}^T \operatorname{AccessCost}(\pi, R_t).$

Due to its mathematical elegance and its connections to many practical applications, GMSSC and its variants have received significant research attention [20, 5, 4, 29]. The special case where the covering requirement is $K(R_t) = 1$ for all requests R_t is known as *Min-Sum Set Cover* (MSSC). MSSC is NP-hard, admits a natural greedy 4-approximation algorithm and is inapproximable in polynomial time within any ratio less than 4, unless P = NP [13]. Approximation algorithms for GMSSC have been considered in a sequence of papers [6, 36, 30] with the state of the art approximation ratio being 12.5. Closing the approximability gap, between 4 and 12.5, for GMSSC remains an interesting open question.

Generalized Min-Sum Set Cover and Online Learning. Virtually all previous work on GMSSC (the recent work of [17] is the only exception) assumes that the algorithm knows the request sequence and the covering requirements well in advance. However, in the practical item ranking setting considered above, one should maintain a high quality ranking online, based on little (if any) information about the favorite items and the demand of new users.

Motivated by that, we study GMSSC as an online learning problem [21]. I.e., we consider a learner that selects permutations over time (without knowledge of future requests), trying to minimize her total access cost, and an adversary that selects requests R_1, \ldots, R_T and their covering requirements, trying to maximize the learner's total access cost. Specifically, at each round $t \geq 1$,

- 1. The learner selects a permutation π_t over the *n* items, i.e., $\pi_t \in [n!]$.
- 2. The adversary selects a request R_t with covering requirement $K(R_t)$.
- 3. The learner incurs a cost equal to $AccessCost(\pi_t, R_t)$.

Based on the past requests R_1, \ldots, R_{t-1} only, an online learning algorithm selects (possibly with the use of randomization) a permutation π_t trying to achieve a total (expected) access cost as close as possible to the total access cost of the optimal permutation π^* . If the cost of the online learning algorithm is at most α times the cost of the optimal permutation, the algorithm is α -regret [21]. If $\alpha = 1$, the algorithm is no-regret. In this work, we investigate the following question:

Question 1. Is there an online learning algorithm for GMSSC that runs in polynomial time and achieves α -regret, for some small constant $\alpha \geq 1$?

Despite a huge volume of work on efficient online learning algorithms and the rich literature on approximation algorithms for GMSSC, Question 1 remains challenging and wide open. Although the *Multiplicative Weights Update* (MWU) algorithm, developed for the general problem of *Learning from Expert Advice*, achieves no-regret for GMSSC, it does not run in polynomial-time. In fact, MWU treats each permutation as a different expert and maintains a weight vector of size n!. Even worse, this is inherent to GMSSC, due to the inapproximability result of [13]. Hence, unless P = NP, MWU's exponential requirements could not be circumvented by a more clever GMSSC-specific implementation, because any polynomial-time α -regret online learning algorithm can be turned into a polynomial-time α -approximation algorithm for GMSSC. Moreover, the results of [32] on obtaining computationally efficient α -regret online learning algorithms from known polynomial time α -approximation algorithms for NP-hard optimization problems do not apply to optimizing non-linear objectives (such as the access cost in GMSSC) over permutations.

Our Approach and Techniques. Departing from previous work, which was mostly focused on black-box reductions from polynomial-time algorithms to polynomial-time online learning algorithms, e.g., [33, 32], we carefully exploit the structure of permutations and GMSSC, and present polynomial-time low-regret online learning deterministic and randomized algorithms for GMSSC, based on dimensionality reduction and Online Projected Gradient Descent.

Our approach consists of two major steps. The first step is to provide an efficient no-regret polynomial-time learning algorithm for a relaxation of GMSSC defined on doubly stochastic matrices. To optimize over doubly stochastic matrices, the learner needs to maintain only n^2 values, instead of the n! values required to directly describe distributions over permutations. This *dimensionality reduction* step allows for a polynomial-time no-regret online algorithm for the relaxed version of GMSSC.

The second step is to provide computationally efficient (deterministic and randomized) online rounding schemes that map doubly stochastic matrices back to probability distributions over permutations. The main



Figure 1: Our general approach, which is independent of the specific variant of GMSSC.

challenge is to guarantee that the expected access cost of the (possibly random) permutation obtained by rounding is within a factor of α from the access cost of the doubly stochastic matrix representing the solution to the relaxed problem. Once such a bound is established, it directly translates to an α -regret online learning algorithm with respect to the optimal permutation for GMSSC. Our approach is summarized in Figure 1. **Designing and Solving the Relaxed Online Learning Problem.** For the relaxed version of GMSSC, we note that any permutation π corresponds to an *integral* doubly stochastic matrix A^{π} , with $A^{\pi}[i, j] = 1$ iff $\pi(j) = i$. Moreover for any request R, each doubly stochastic matrix is associated with a *fractional access* cost. For integral doubly stochastic matrices, the fractional access cost is practically identical to the access

The fractional access cost is given by the optimal solution of an (exponentially large) configuration linear program (LP) that relaxes GMSSC to doubly stochastic matrices (see also [30]), and is a convex function. Thus, we can use *Online Projected Gradient Descent* (OPGD) [44] to produce a *no-regret* sequence of doubly stochastic matrices for the GMSSC relaxation. However, the efficient computation of the subgradient is far from trivial, due to the exponential size of the configuration LP. A key technical step is to show that the subgradient of the configuration LP can be computed in polynomial time, by solving its dual (which is of exponential size, so we resort to the elipsoid method and use an appropriate separation oracle).

cost of GMSSC in the respective permutation.

Our Results. In nutshell, we resolve Question 1 in the affirmative. In addition to solving the relaxed version of GMSSC by a polynomial-time no-regret online learning algorithm, as described above, we present a *polynomial-time randomized rounding scheme* that maps any doubly stochastic matrix to a probability distribution on permutations. The expected access cost of such a probability distribution is at most 28 times the fractional access cost of the corresponding doubly stochastic matrix. Consequently, a 28-regret *polynomial-time randomized online learning algorithm* for GMSSC can be derived by applying, in each round, this rounding scheme to the doubly stochastic matrix A^t , produced by OPGD. For the important special case of MSSC, we improve the regret bound to 11.713 via a similar randomized rounding scheme that exploits the fact that K(R) = 1 for all requests.

We also present a polynomial-time deterministic rounding scheme mapping any (possibly fractional) doubly stochastic matrix to permutations. As before, applying this scheme to the sequence of doubly stochastic matrices produced by OPGD for the relaxation of GMSSC leads to a polynomial-time deterministic online learning algorithm with regret $2 \max_t |R_t|$ for MSSC. Such a nontrivial upper bound on the regret of deterministic online learning algorithms is rather surprising. Typically, learners that select their actions deterministically fail to achieve any nontrivial regret bounds (e.g., recall that in Learning From Expert Advice, any deterministic online algorithm has $\Omega(\#\text{experts})$ regret, which in case of MSSC is n!). Although $2 \max_t |R_t|$ is not constant, one should expect that the requests are rather small in most practical applications. The above result is approximately tight, since any deterministic online learning algorithm must have regret

at least $\max_t |R_t|/2$ [17, Theorem 1.1]. We should also highlight that the positive results of [17] do not imply the existence of computationally efficient online learning algorithms for MSSC, because their approach is based on the MWU algorithm and uses a state space of n!. The state of the art and our results (in bold) are summarized below.

	Running Time	Upper Bound (Regret)	Lower Bound (Regret)	
GMSSC	Exponential (MWU)	1	1	
GMSSC	Polynomial	28	4 (any polynomial time)	
MSSC	Polynomial	11.713	4 (any polynomial time)	
MSSC	Exponential (deterministic)	$2 \cdot \max_t R_t $	$\frac{\max_t R_t }{2}$ (any deterministic)	
MSSC	Polynomial (deterministic)	$2 \cdot \max_t R_t $	$\frac{\max_t R_t }{2}$ (any deterministic)	

Related Work. Our work relates with the long line of research concerning the design of time-efficient online learning algorithms in various combinatorial domains in which the number of possible actions is exponentially large. Such domains include online routing [26, 3], selection of permutations [40, 42, 2, 27], selection of binary search trees [41], submodular minimization/maximization [23, 31, 37], matrix completion [24], contextual bandits [1, 11] and many more.

Apart from the above line of works, concerning the design of time-efficient online learning algorithms in specific settings, another line of research studies the design of online learning algorithms considering *black-box access* in offline algorithms [33, 35, 39, 34, 15, 10, 25, 32, 18, 19, 22]. In their seminal work [33], Kalai et al. showed how a polynomial-time algorithm solving optimally the underlying combinatorial problem, can be converted into a no-regret polynomial-time online learning algorithm. The result of Kalai et al. was subsequently improved [35, 39, 34] for settings in which the underlying problem can be (optimally) solved by a specific approach, such as dynamic programming. Although there do not exist such general reductions for α -approximation (offline) algorithms (without taking into account the combinatorial structure of each specific setting [25]), Kakade et al. presented such a reduction for the (fairly general) class of *linear optimization problems* [32]. Their result was subsequently improved by [18, 19, 22]. We remark that the above results do not apply in our setting since GMSSC can neither be optimally solved in polynomial-time nor is a *linear optimization problem*.

Finally our works also relates with a recent line of research studying time-efficient online learning algorithms in settings related to selection of permutations and rankings [42, 2, 27, 38, 43]. The setting considered in [42, 2, 27] is very similar to GMSSC with the difference that once request R_t is revealed, the learner pays the sum of the positions of R_t 's elements in permutation π_t . In this case the underlying combinatorial optimization problem can be solved in polynomial-time meaning that the reduction of [33] produces a time-efficient no-regret online learning algorithm. As a result, all the above works focus on improving the vanishing rate of time-average regret. The setting considered in [38] is based on the submodular maximization problem. In particular, the number of available positions is less than the number of elements, while the cost of the selected assignment depends on the set of elements assigned to the slots (their order does not matter). Although this problem is NP-hard, it admits an (1 - 1/e)-approximation algorithm which is matched by the presented online learning algorithm. Finally in [43], the cost of the selected permutation is its distance from a permutation selected by the adversary. In this case the underlying combinatorial optimization problem admits an offline 11/9-approximation algorithm, while a polynomial-time online learning algorithm with 3/2-regret is presented. We note that GMSSC admits a fairly more complicated combinatorial structure from the above settings and this is indicated by its 4 inapproximability result.

2 Definitions and Notation

Definition 1 (Subgradient). Given a function $f : D \mapsto \mathbb{R}$, with $D \subseteq \mathbb{R}^n$, a vector $g \in \mathbb{R}^n$ is a subgradient of f at point $x \in \mathbb{R}^n$, denoted $g \in \partial F(x)$, if $f(y) \ge f(x) + g^{\top}(y - x)$, for all $y \in D$.

A matrix $A \in [0,1]^{n \times n}$ is *doubly stochastic*, if (i) $A_{ij} \ge 0$, for all $1 \le i, j \le n$, (ii) $\sum_{i=1}^{n} A_{ij} = 1$, for all $1 \le j \le n$, and (iii) $\sum_{j=1}^{n} A_{ij} = 1$, for all $1 \le i \le n$. We let DS denote the set of $n \times n$ doubly stochastic matrices.

Any permutation $\pi \in [n!]$ can be represented by an *integral* doubly-stochastic A^{π} , where $A_{ij}^{\pi} = 1$ iff

 $\pi(j) = i$. Under this representation, the access cost of GMSSC, defined in (1), becomes:

$$\operatorname{AccessCost}(\pi, R) = \sum_{i=1}^{n} \min\left\{1, \left(\operatorname{K}(R) - \sum_{j=1}^{i-1} \sum_{e \in R} A_{ej}^{\pi}\right)_{+}\right\}, \qquad (2)$$

where we define $(x - y)_{+} = \max\{x - y, 0\}.$

A key notion for our algorithms and analysis is that of *configurations*. Given a request $R \subset U$, a *configuration* F is an assignment of the elements $e \in R$ to positions $j \in [n]$ such that no two elements $e, e' \in R$ share the same position. Intuitively, a configuration wrt. a request R is the set of all permutations $\pi \in [n!]$ with the elements of R in the exact same positions as indicated by F. As a result, all permutations $\pi \in [n!]$ that agree with a configuration F wrt. a request R have the same AccessCost (π, R) . In the following, F(R) denotes the set of all configurations wrt. a request R and C_F denotes the access cost AccessCost (π, R) of any permutation $\pi \in [n!]$ that agrees with the configuration $F \in F(R)$.

Example 1. Let $R = \{2, 5, 7\}$ with K(R) = 2. The configuration $F_1 = \{(2, 3), (5, 1), (7, 10)\}$ stands for the set of permutations $\pi \in [n!]$ in which (i) $\pi(3) = 2$, (ii) $\pi(1) = 5$, and (iii) $\pi(10) = 7$. The configuration F_1 is valid (i.e., $F_1 \in F(R)$), because no elements of R share the same position. Moreover, $C_{F_1} = 3$, because any permutation π agreeing with F has cost 3 for K(R) = 2. Similarly, for the configuration $F_2 = \{(2,3), (5,1), (7,2)\}, C_{F_2} = 2$.

3 Solving a Relaxation of Generalized Min-Sum Set Cover

Next, we present an online learning problem for a relaxed version of GMSSC in the space of doubly stochastic matrices. Specifically, we consider an online learning setting where, in each round $t \ge 1$,

- 1. The learner selects a doubly stochastic matrix $A^t \in DS$.
- 2. The adversary selects a request R_t with covering requirements $K(R_t)$.
- 3. The learner incurs the fractional access cost $FAC_{R_t}(A^t)$ presented in Definition 2.

Definition 2 (Fractional Access Cost). Given a request R with covering requirements K(R), the fractional access cost of a doubly stochastic matrix A, denoted as $FAC_R(A)$ is the value of the following linear program:

minimize
$$\sum_{F \in F(R)} C_F \cdot y_F + \frac{n^4}{\epsilon} \cdot \sum_{e \in R} \sum_{j=1}^n |A_{ej} - \sum_{F:(e,j) \in F} y_F|$$
subject to
$$\sum_{F \in F(R)} y_F = 1$$
$$y_F \ge 0, \ \forall F \in F(R)$$
(FLP)

We always assume a fixed accuracy parameter ϵ (see also Theorem 1 about the role of ϵ). Hence, for simplicity, we always ignore the dependence of $FAC_R(A)$ on ϵ . We should highlight that we need to deviate from the configuration LP of [30, Sec. 2], because *OPGD* requires an upper bound in the subgradient's norm. The n^4 term in (FLP) was appropriately selected so as to ensure that the *access cost* of the probability distribution on permutations produced by a doubly stochastic matrix is upper bounded by its *fractional access cost* (see Section B.1).

An important property of the fractional access cost in Definition 2 is that for all integral doubly stochastic matrices, it is bounded from above by the access cost of GMSSC in (2). For that, simply note that a feasible solution is setting $y_F = 1$ only for the configuration that "agrees" in the resources of R with the permutation of the integral matrix A.

Corollary 1. For any integral doubly stochastic matrix A^{π} corresponding to a permutation $\pi \in [n!]$,

 $\operatorname{FAC}_R(A^{\pi}) \leq \operatorname{AccessCost}(\pi, R).$

For $A^1, A^2 \in DS$, it is $FAC_{R_t} (\lambda A^1 + (1 - \lambda)A^2) \leq \lambda \cdot FAC_{R_t} (A^1) + (1 - \lambda) \cdot FAC_{R_t} (A^2)$, meaning that $FAC_{R_t}(\cdot)$ is a convex function in the space of doubly stochastic matrices. Since doubly stochastic matrices form a convex set, *Online Projected Gradient Descent* [44] is a *no-regret* online learning algorithm for the relaxed version of GMSSC.

3.1Implementing Online Gradient Descent in Polynomial-time

Online Gradient Descent requires, in each round t, the computation of a subgradient of the fractional access cost $FAC_{R_t}(A_t)$ (see also Definition 1). Specifically, given a request R and a doubly stochastic matrix A, a vector $g \in \mathbb{R}^{n^2}$ belongs to the subgradient $\partial FAC_R(A)$, if for any $B \in DS$,

$$FAC_R(B) \ge FAC_R(A) + g^{\top}(B - A), \qquad (3)$$

where we slightly abuse the notation and think of matrices A and B as vectors in $[0, 1]^{n^2}$.

Computing a subgradient $g \in \partial FAC_R(A)$ in polynomial-time is far from trivial, because the fractional access cost $FAC_R(A)$ does not admit a closed form, since its value is determined by the optimal solution to (FLP). Moreover, (FLP) has exponentially many variables y_F , one for each configuration $F \in F(R)$. We next show how to compute a subgradient $q \in \partial FAC_R(A)$ by using linear programming duality and solving the dual of (FLP), which is presented below:

maximize
$$\lambda + \sum_{e \in R} \sum_{j=1}^{n} A_{ej} \cdot \lambda_{ej}$$

subject to $\lambda + \sum_{(e,j) \in F} \lambda_{ej} \leq C_F$, for all $F \in F(R)$
 $|\lambda_{ej}| \leq n^4/\epsilon$ (4)

Lemma 1. For any request R and any stochastic matrix $A \in DS$, let $g \in \mathbb{R}^{n^2}$ denote the vector consisting of the n^2 values of the variables $\lambda_{e_i}^*$ in the optimal solution of (4). Then, for any $B \in DS$,

$$\operatorname{FAC}_R(B) \ge \operatorname{FAC}_R(A) + g^{\top}(B - A)$$

Moreover the Euclidean norm of g is upper bounded by n^5/ϵ , i.e., $\|g\|_2 \leq n^5/\epsilon$.

Lemma 1 shows that a subgradient $g \in \partial FAC_R(A)$ can be obtained from the solution to the dual LP (4). Although (4) has exponentially many constraints, we can solve it in polynomial-time by the ellipsoid method, through the use of an appropriate separation oracle.¹ In fact, our separation oracle results from a simple modification of the separation oracle in [30, Sec. 2.3] (see also Section A). Now, the reasons for the particular form of fractional access cost in Definition 2 become clear: (i) it allows for efficient computation of the subgradients, and (ii) the dual constraints $|\lambda_{ej}| \leq n^4/\epsilon$ imply that the subgradient's norm is always bounded by n^5/ϵ .

Remark 1. For the Min-Sum Set Cover problem, the use of the ellipsoid method (for the computation of the subgradient vector) can be replaced by a more efficient quadratic-time algorithm (see Appendix B.2).

Having established polynomial-time computation for the subgradients, Online Projected Gradient Descent takes the form of Algorithm 1 in our specific setting.

Algorithm 1	Online Proje	ected Gradient	Decent in Dou	ubly Stochast	ic Matrices
				•/	

- 1: Initially, the player selects the matrix $A^1 = 1/n \cdot 1_{n \times n}$.
- 2: for all rounds $t = 1 \cdots T$ do
- 3: The adversary selects a request $R_t \subseteq U$ with covering requirements $K(R_t)$.
- The learner receives cost, $FAC_{R_t}(A^t)$. 4:
- The learner computes a subgradient $g_t \in \partial FAC_{R_t}(A^t)$ by solving the dual of (FLP). 5:
- 6:
- The learner computes the matrix, $\hat{A} = A^t 2\epsilon \cdot g_t / (n^{4.5}\sqrt{t})$. The learner adopts the matrix, $A^{t+1} = \arg \min_{A \in DS} ||A \hat{A}||_{\rm F}$ 7:
- 8: end for

¹Interestingly, GMSSC seems to be the most general version of min-sum-set-cover-like ranking problems that allow for an efficient subgradient computation through the dual of the configuration LP (FLP). E.g., for the version of Min-Sum-Set-Cover with submodular costs considered in [4], determining the feasibility of a potential solution to (4) is NP-hard. This is true even for very special case where the cover time function used in [4] is additive.

Step 6 of Algorithm 1 is the gradient step. In Online Projected Gradient Descent, this step is performed with step-size $D/(G\sqrt{t})$, where D and G are upper bounds on the diameter of the action space and on the Euclidean norm of the subgradients. In our case, the action space is the set of doubly stochastic matrices. Since $\max_{A,B\in DS} ||A-B||_F \leq 2\sqrt{n}$ the parameter $D = 2\sqrt{n}$, and $G = n^5/\epsilon$, by Lemma 1. Hence, our step-size is $2\epsilon/(n^{4.5}\sqrt{t})$. The projection step (Step 7) is implemented in polynomial-time, because projecting to doubly stochastic matrices is a convex problem [16]. We conclude the section by plugging in the parameters $G = n^5/\epsilon$ and $D = 2\sqrt{n}$ to the regret bounds of Online Projected Gradient Descent [44], thus obtaining Theorem 1.

Theorem 1. For any $\epsilon > 0$ and any request sequence R_1, \ldots, R_T , the sequence of doubly stochastic matrices A^1, \ldots, A^T produced by Online Projected Gradient Descent (Algorithm 1) satisfies, $\frac{1}{T} \sum_{t=1}^T \text{FAC}_{R_t}(A^t) \leq \frac{1}{T} \min_{A \in \text{DS}} \sum_{t=1}^T \text{FAC}_{R_t}(A) + O\left(\frac{n^{5.5}}{\epsilon\sqrt{T}}\right).$

4 Converting Doubly Stochastic Matrices to Distributions on Permutations

Next, we present polynomial-time rounding schemes that map a doubly stochastic matrix back to a probability distribution on permutations. Our schemes ensure that the resulting permutation (random or deterministic) has access cost at most α times the fractional access cost of the corresponding doubly stochastic matrix. Combining such schemes with Algorithm 1, we obtain polynomial-time α -regret online learning algorithms for GMSSC.

Due to lack of space, we only present the deterministic rounding scheme, which is intuitive and easy to explain. Most of its analysis and the description of the randomized rounding schemes are deferred to the supplementary material.

Algorithm 2 Converting Doubly Stochastic Matrices to Permutations

Input: A doubly stochastic matrix $A \in DS$, a parameter r and a parameter $\alpha > 0$. **Output:** A deterministic permutation $\pi_A \in [n!]$.

- 1: Rem $\leftarrow \{1, \ldots, n\}$
- 2: for k = 1 to $\lfloor n/r \rfloor$ do
- 3: Let R_k be any $(1 + \alpha)$ -approximate solution to the following problem:

$$\min_{\substack{R\subseteq \operatorname{Rem}: |R|=r}} \sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i-1} \sum_{e \in R} A_{ej} \right)_{+}$$

- 4: Assign the elements of R_k to positions $(k-1) \cdot r + 1, \ldots, k \cdot r$ of π_A in any order.
- 5: $\operatorname{Rem} \leftarrow \operatorname{Rem} \setminus R_k$
- 6: end for
- 7: **return** the resulting permutation $\pi_A \in [n!]$.

Algorithm 2 aims to produce a permutation $\pi_A \in [n!]$ from the doubly stochastic matrix A such that the AccessCost(π_A, R) is approximately bounded by $\text{FAR}_R(A)$ for any request R with $|R| \leq r$ and K(R) = 1. Algorithm 2 is based on the following intuitive greedy criterion:

Assign to the first r available positions of π_A the elements of the request of size r with minimum fractional cost of Definition 2 wrt. the doubly stochastic matrix A. Then, remove these elements and repeat.

Unfortunately the greedy step above involves the solution to an NP-hard optimization problem. Nevertheless, we can approximate it with an FPTAS (Fully Polynomial-Time Approximation Scheme). The $(1+\alpha)$ -approximation algorithm used in Step 3 of Algorithm 2 runs in $\Theta(n^4r^3/\alpha^2)$ and is presented and analyzed in Section B.5. Theorem 2 (proved in Section B.3) summarizes the guarantees on the access cost of a permutation π_A produced by Algorithm 2. **Theorem 2.** Let π_A denote the permutation produced by Algorithm 2 when the doubly stochastic matrix A is given as input. Then for any request R with K(R) = 1 and $|R| \leq r$,

$$\operatorname{AccessCost}(\pi_A, R) \leq 2(1+\epsilon)(1+\alpha)^2 r \cdot \operatorname{FAC}_R(A),$$

with $\epsilon > 0$ as in Definition 2. Moreover, Step 3, can be implemented in $\Theta(n^4 r^3/\alpha^2)$ steps.

We now show how Algorithm 1 and Algorithm 2 can be combined to produce a polynomial-time deterministic online learning algorithm for MSSC with regret roughly $2 \max_{1 \le t \le T} |R_t|$. For any adversarially selected sequence of requests R_1, \ldots, R_T with $K(R_t) = 1$ and $|R_t| \le r$, the learner runs Algorithm 1 in the background, while at each round t uses Algorithm 2 to produce the permutation π_{A^t} by the doubly stochastic matrix $A^t \in DS$. Then,

$$\frac{1}{T} \sum_{t=1}^{T} \operatorname{AccessCost}(\pi_{A^{t}}, R_{t}) \leq \frac{1}{T} \cdot \sum_{t=1}^{T} 2(1+\epsilon)(1+\alpha)^{2} r \cdot \operatorname{FAC}_{R_{t}}(A^{t}) \\
\leq \frac{2r}{T}(1+\epsilon)(1+\alpha)^{2} \cdot \min_{A \in \mathrm{DS}} \sum_{t=1}^{T} \operatorname{FAC}_{R^{t}}(A) + O\left(\frac{n^{5.5}}{\epsilon\sqrt{T}}\right) \\
\leq \frac{2r}{T}(1+\epsilon)(1+\alpha)^{2} \cdot \sum_{t=1}^{T} \operatorname{FAC}_{R^{t}}(A^{\pi^{*}}) + O\left(\frac{n^{5.5}}{\epsilon\sqrt{T}}\right) \\
\leq \frac{2r}{T}(1+\epsilon)(1+\alpha)^{2} \cdot \sum_{t=1}^{T} \operatorname{AccessCost}(\pi^{*}, R^{t}) + O\left(\frac{n^{5.5}}{\epsilon\sqrt{T}}\right)$$

The first inequality follows by Theorem 2, the second by Theorem 1 and the last by Corollary 1.

Via the use of randomized rounding schemes we can substantially improve both on the assumptions and the guarantee of Theorem 2. Algorithm 3 (presented in Section B.1), describes such a scheme that converts any doubly stochastic matrix A to a probability distribution over permutations, while Theorem 3 (also proven in Section B.1) establishes an approximation guarantee (arbitrarily) close to 28 on the expected access cost.

Theorem 3. Let P_A denote the probability distribution over permutations that Algorithm 3 produces given as input an $A \in DS$. For any request R,

$$\mathbb{E}_{\pi \sim \mathcal{P}_{\mathcal{A}}}[\operatorname{AccessCost}(\pi, R)] \le 28(1+\epsilon) \cdot \operatorname{FAC}_{R}(A)$$

where $\epsilon > 0$ is the parameter used in Definition 2.

Using Theorem 3 instead of Theorem 2 in the previously exhibited analysis, implies that combining Algorithms 1 and 3 leads to a *polynomial-time randomized online learning algorithm* for GMSSC with $28(1 + \epsilon)$ regret.

In Section B.2 we improve Theorem 3 for the special case of MSSC. The randomized rounding scheme described in Algorithm 4 admits the approximation guarantee of Theorem 4, which implies a *polynomial-time* randomized online learning algorithm for MSSC with $11.713(1 + \epsilon)$ regret

Theorem 4. Let P_A denote the probability distribution over permutations that Algorithm 4 produces given as input an $A \in DS$. For any request R with covering requirement K(R) = 1,

$$\mathbb{E}_{\pi \sim \mathcal{P}_A}[\operatorname{AccessCost}(\pi, R)] \le 11.713(1+\epsilon) \cdot \operatorname{FAC}_R(A)$$

where $\epsilon > 0$ is the parameter used in Definition 2.

5 Experimental Evaluations

In this section we provide experimental evaluations of all the proposed online learning algorithms (both deterministic and randomized) for *Min-Sum Set Cover*. Surprisingly enough our simulations seem to

suggest that the *deterministic rounding scheme* proposed in Algorithm 2, performs significantly better than its theoretical guarantee, stated in Theorem 2, that associates its regret with the cardinality of the sets. The following figures illustrate the performance of Algorithm 2 and Algorithm 4, and compare it with the performance of the offline algorithm proposed by Feige et al. [13] and the performance of selecting a permutation uniformly at random at each round. In the left figure each request contains either element 1 or 2 and four additional randomly selected elements, while in the right figure each request contains one of the elements $\{1, 2, 3, 4, 5\}$ and nine more randomly selected elements.² We remark that in our experimental evaluations, we solve the optimization problem of Step 3 in Algorithm 2 through a simple heuristic that we present in Appendix B.6, while for the computation of the subgradients we use the formula presented in Corollary 3. The code used for the presented simulations can be found at https://github.com/sskoul/ID2216.



6 Conclusion

This work examines polynomial-time online learning algorithms for *(Generalized) Min-Sum Set Cover.* Our results are based on solving a relaxed online learning problem of smaller dimension via *Online Projected Gradient Descent*, the solution of which is transformed at each round into a solution of the initial action space with bounded increase in the cost. To do so, the cost function of the relaxed online learning problem is defined by the value of a linear program with exponentially many constraints. Despite its exponential size, we show that the subgradients can be efficiently computed via associating them with the variables of the LP's *dual.* We believe that the bridge between online learning algorithms (e.g. online projected gradient descent) and traditional algorithmic tools (e.g. duality, separation oracles, deterministic/randomized rounding schemes), introduced in this work, is a promising new framework for the design of efficient online learning algorithms in high dimensional combinatorial domains. Finally closing the gap between our regret bounds and the lower bound of 4, which holds for polynomial-time online learning algorithms for MSSC, is an interesting open problem.

Broader Impact

We are living in a world of abundance, where each individual is provided myriad of options in terms of available products and services (e.g. music selection, movies etc.). Unfortunately this overabundance makes the cost of exploring all of them prohibitively large. This problem is only compounded by the fast turn around of new trends at a seemingly ever increasing rate. Our algorithmic techniques provide a practically applicable methodology for managing this complexity.

²In the subsequent figures the curves describing the performance of each algorithm are placed in the following top-down order i) *Selecting a permutation uniformly at random*, ii) *Algorithm* 2, iii) *Algorithm* 4 and iv) Feige-Lovasz-Tetali algorithm [13].

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Supplementary Material Efficient Online Learning of Optimal Rankings: Dimensionality Reduction via Gradient Descent

A Omitted Proofs of Section 3

Proof of Lemma 1. To simplify notation, let $\lambda^*(A)$, $\lambda^*_{ej}(A)$ denote the values of the variables $\lambda^*(A)$, $\lambda^*(A)_{ej}$ in the optimal solution of the dual program written with respect to doubly stochastic matrix $A \in DS$. Respectively $\lambda^*(B)$, $\lambda^*_{ej}(B)$ for the doubly stochastic matrix $B \in DS$. By strong duality, we have that

$$\operatorname{FAC}_R(A) = \lambda^*(A) + \sum_{e \in R} \sum_{j=1}^n A_{ej} \cdot \lambda^*_{ej}(A) \text{ and } \operatorname{FAC}_R(B) = \lambda^*(B) + \sum_{e \in R} \sum_{j=1}^n B_{ej} \cdot \lambda^*_{ej}(B)$$

Since matrices A and B only affect the objective function of the dual and not its constraints, the solution $\lambda^*(A), \lambda_{e_j}^*(A)$ is a feasible solution for the dual program written according to matrix B. By the optimality of $\lambda^*(B), \lambda_{e_j}^*(B)$ we get,

$$\operatorname{FAC}_R(B) = \lambda^*(B) + \sum_{e \in R} \sum_{j=1}^n B_{ej} \cdot \lambda^*_{ej}(B) \ge \lambda^*(A) + \sum_{e \in R} \sum_{j=1}^n B_{ej} \cdot \lambda^*_{ej}(A)$$

As a result, we get that $FAC_R(B) - FAC_R(A) \ge \sum_{e \in R} \sum_{j=1}^n \lambda_{ej}^*(A) \cdot (B_{ej} - A_{ej})$ implying that the vector g containing the $\lambda_{ej}^*(A)$'s, is a subgradient of $FAC_R(\cdot)$ at point A, i.e., $g \in \partial FAC_R(A)$. The inequality $\|g\|_2 \le n^5/\epsilon$ directly follows by the fact that $|\lambda^*(A)_{ej}| \le n^4/\epsilon$.

Separation Oracle for the LP in Equation 4: The dual linear program of (4) is differs from the LP_{dual} in [30, Sec. 2.2] only in the constraints $|\lambda_{ej}| \leq n^4/\epsilon$, which are only present in (4). [30, Sec. 2.2] present a separation oracle for their LP_{dual} (i.e., for (4), without the constraints $|\lambda_{ej}| \leq n^4/\epsilon$), which is based on formulating and solving a min-cost flow problem. Since, in case of (4), the we have only n^2 additional constraints $|\lambda_{ej}| \leq n^4/\epsilon$, we can first check whether these constraints are satisfied by the current solution and then run the separation oracle of [30].

B Omitted Proofs of Section 4

B.1 Proof of Theorem 3

In Algorithm 3, we present the online randomized rounding scheme that combined with Projected Gradient Descent (Algorithm 1) produces a polynomial-time randomized online learning algorithm for GMSSC with (roughly) 28 regret. The randomized rounding scheme described in Algorithm 3 was introduced by [36] to provide a 28-approximation algorithm for the (offline) GMSSC. [36] proved that this randomized rounding scheme produces a random permutation with access cost at most 28 times greater than the optimal fractional value of the LP relaxation of GMSSC introduced in [6]. We remark that this LP relaxation cannot be translated to an equivalent relaxed online learning problem as the one we formulated using the fractional access cost of Definition 2. The goal of the section is to prove Theorem 3 which extends the result of [36] to the fractional access cost of Definition 2.

Algorithm 3 Converting Doubly Stochastic Matrices to Probability Distributions over Permutations

Input: A doubly stochastic matrix $A \in DS$. **Output:** A probability distribution over permutations, $P_A \sim \pi \in [n!]$ 1: Randomly pick $\alpha \in (0, 1)$ with probability density function $f(\alpha) = 2\alpha$. 2: Set $B \leftarrow (5.03/\alpha) \cdot A$ 3: for all elements e = 1 to n do for all positions j = 1 to $\lfloor n/2 \rfloor$ do 4: $B_{e,2j} \leftarrow B_{e,2j} + B_{e,j}$. 5: end for 6: 7: end for 8: for all elements e = 1 to n do Pick α_e uniformly at random in [0, 1]. 9: Find the effective index $i_e^{\alpha} \leftarrow \arg \max_i \{i : \sum_{j=1}^{i-1} B_{ej} < \alpha_e\}.$ 10: 11: end for 12: Output the elements according to the order of i_e 's.

Definition 3. For a request R with covering requirements K(R), we define the cost $SW_R : DS \mapsto \mathbb{R}$ on the doubly stochastic matrices as follows: For any doubly stochastic matrix $A \in DS$, the value $SW_R(A)$ equals the value of the following linear program,

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} \left(1 - z_{i}\right) \\ \text{subject to} & \left(\mathrm{K}(R) - |M|\right) \cdot z_{i} \leq \sum_{j=1}^{i-1} \sum_{e \in R \setminus M} A_{ej} \quad \text{for all} \quad M \subseteq R \\ & z_{i} \in [0, 1] \quad \text{for all } 1 \leq i \leq n \end{array}$$

Lemma 2. [36] For any doubly stochastic matrix $A \in DS$,

$$\mathbb{E}_{\pi \sim \mathcal{P}_A}[\operatorname{AccessCost}(\pi, R)] \le 28 \cdot \operatorname{SW}_R(A)$$

where P_A is the probability distribution over the permutation produced by Algorithm 3 when the matrix A was given as input.

In Lemma 3 we associate the cost $SW_R(\cdot)$ of Definition 3 with the *fractional access cost* $FAC_R(\cdot)$ of Definition 2. Then Theorem 3 directly follows by Lemma 2 and Lemma 3.

Lemma 3. For any doubly stochastic matrix $A \in DS$,

$$SW_R(A) \le (1+\epsilon) \cdot FAC_R(A)$$

where $\epsilon > 0$ is the parameter of the linear program (FLP) in Definition 2.

Proof. Starting from the optimal solution y_F of the linear program (FLP) of $FAC_R(A)$ in Definition 2, we construct a feasible solution for the linear program of $SW_R(A)$ of Definition 3 with cost approximately bounded by $(1 + \epsilon) \cdot FAC_R(A)$. We first prove Claim 1 that is crucial for the subsequent analysis.

Claim 1. For any element $e \in R$ and position $1 \le j \le n$, $|A_{ej} - \sum_{F:(e,je,e,j)\in F} y_F^*| \le \epsilon/n^3$.

Proof. Since A is a doubly stochastic matrix, by the Birkhoff-von Neumann theorem there exists a vector \hat{y} with $\hat{y}_F \ge 0$ and $\sum_{F \in F(R)} \hat{y}_F = 1$ such that

$$|A_{ej} - \sum_{F:(e,j)\in F} \hat{y}_F| = 0 \text{ for all } e \in R \text{ and } 1 \le j \le n$$

Since y^* is the optimal solution, we have that

$$\sum_{F \in \mathcal{F}(R)} C_F \cdot y_F^* + \frac{n^4}{\epsilon} \cdot \sum_{e \in R} \sum_{j=1}^n |A_{ej} - \sum_{F:(e,j) \in F} y_F^*| \le \sum_{F \in \mathcal{F}(R)} C_F \cdot \hat{y}_F.$$

Now the claim follows by the fact that $1 \leq C_F \leq n$, $\hat{y}_F \geq 0$ and $\sum_{F \in F(R)} \hat{y}_F = 1$.

Having established Claim 1, we construct the solution z^* that is feasible for the linear program of Definition 3 and its value (under the linear program of Definition 3), is upper bounded by $(1 + \epsilon) \cdot FAC_R(A)$. For each position $1 \le i \le n$,

$$z_i^* = \left(\sum_{F \in \mathcal{F}(R): C_F \le i-1} y_F^* - \frac{\epsilon}{n}\right)_+$$

We first prove that z^* is feasible for the linear program of Definition 3. At first observe that in case $z_i^* = 0$ or $\mathcal{K}(R) - |M| \leq 0$ for some $M \subseteq R$, the constraint $(\mathcal{K}(R) - |M|) \cdot z_i \leq \sum_{j=1}^{i-1} \sum_{e \in R \setminus M} A_{ej}$ is trivially satisfied. We thus turn our attention in the cases where $z_i^* = \sum_{F:C_F \leq i-1} y_F^* - \epsilon/n > 0$ and $\mathcal{K}(R) - |M| \geq 1$ (recall, $\mathcal{K}(R)$ and |M| are integers). Applying Claim 1 we get that,

$$\sum_{e \in R \setminus M} \sum_{j=1}^{i-1} A_{ej} \geq \sum_{e \in R \setminus M} \sum_{j=1}^{i-1} \left(\sum_{F:(e,j) \in F} y_F^* - \epsilon/n^3 \right)$$

$$\geq \sum_{e \in R \setminus M} \sum_{j=1}^{i-1} \sum_{F:(e,j) \in F} y_F^* - \epsilon/n$$

$$= \sum_{F \in F(R)} y_F^* \sum_{e \in R \setminus M} \sum_{j=1}^{i-1} \mathbf{1}[(e,j) \in F] - \epsilon/n$$

$$\geq \sum_{F:C_F < i} y_F^* \sum_{e \in R \setminus M} \sum_{j=1}^{i-1} \mathbf{1}[(e,j) \in F] - \epsilon/n$$

$$\geq (K(R) - |M|) \sum_{F:C_F < i} y_F^* - \epsilon/n$$

$$= (K(R) - |M|) \cdot z_i^* + \epsilon \frac{K(R) - |M|}{n} - \epsilon/n$$

$$\geq (K(R) - |M|) \cdot z_i^*$$

where the second to last inequality follows from $C_F < i$, and the last equation and the last inequality follows from $z_i^* + \epsilon/n = \sum_{F:C_F \leq i-1} y_F^*$ and $K(R) - |M| \geq 1$, respectively.

We complete the proof of Lemma 3 by showing that $\sum_{i=1}^{n} (1 - z_i^*) \leq (1 + \epsilon) \cdot \text{FAC}_R(A)$.

$$SW_R(A) \leq \sum_{i=1}^n (1-z_i^*)$$

$$\leq \sum_{i=1}^n \left(1 - \sum_{F:C_F < i} y_F^* + \epsilon/n\right)$$

$$= \sum_{i=1}^n \left(1 - \sum_{F:C_F < i} y_F^*\right) + \epsilon$$

$$= \sum_{i=1}^n \sum_{F:C_F \ge i} y_F^* + \epsilon$$

$$= \sum_{F \in F(R)} C_F \cdot y_F^* + \epsilon$$

$$\leq (1+\epsilon) \cdot \operatorname{FAC}_R(A)$$

B.2 Proof of Theorem 4

We first present the online sampling scheme, described in Algorithm 4, that produces the 11.713 guarantee of Theorem 4.

Algorithm 4 Converting Doubly Stochastic Matrices to Probability Distribution (the case of MSSC) **Input:** A doubly stochastic matrix $A \in DS$. **Output:** A probability distribution over permutations, $P_A \sim \pi \in [n!]$. 1: Randomly pick $\alpha \in (0, 1)$ with probability density function $f(\alpha) = 2\alpha$. 2: Set $B \leftarrow Q \cdot A$ where $Q \leftarrow 1.6783/\alpha$. 3: for all elements e = 1 to n do for all positions j = 1 to |n/2| do 4: $B_{e,2j} \leftarrow B_{e,2j} + B_{e,j}$ 5: end for 6: 7: end for 8: for all elements e = 1 to n do Pick α_e uniformly at random in [0, 1]. $i_e \leftarrow \max\{i : \sum_{j=1}^{i-1} B_{ej} < \alpha_e\}$ 9: 10: 11: end for 12: Output the elements according to the order of i_e 's.

We dedicate the rest of the section to prove Theorem 4. Notice that Algorithm 4 is identical to Algorithm 3 with a slight difference in Step 2. Taking advantage of K(R) = 1, with tailored analysis, we significantly improve to 11.713 the 28 bound of Lemma 2. Once Lemma 4 below is established, Theorem 4 follows by the exact same steps that Theorem 3 follows using Lemma 2. The proof of Lemma 4 is concluded at the end of the section.

Lemma 4. Let P_A denote the probability distribution over permutations produced by Algorithm 4 when matrix A is given as input. For all requests R with K(R) = 1,

$$\mathop{\mathbb{E}}_{\pi \sim \mathcal{P}_A} [\operatorname{AccessCost}(\pi, R)] \le 11.713 \cdot \operatorname{SW}_R(A)$$

where $SW_R(\cdot)$ is the cost of Definition 3.

In fact $SW_R(\cdot)$ takes a simpler form.

Corollary 2. For any request R with covering requirement K(R) = 1, the cost $SW_R(\cdot)$ of Definition 3 takes the following simpler form,

$$SW_R(A) = \sum_{i=1}^n \left(1 - \sum_{j=1}^{i-1} \sum_{e \in R} A_{ej} \right)$$

Lemma 5. [36] For the matrix B constructed at Step 2 of Algorithm 4, the following holds:

1. $\sum_{j=1}^{2^{k}i} B_{ej} \ge (k+1) \sum_{j=1}^{i} A_{ej}$ 2. $\sum_{j=1}^{i} \sum_{e=1}^{n} B_{ej} \le 2Q \cdot i.$

Condition 2 of Lemma 5 allows for a bound on the expected access cost of the probability distribution produced by Algorithm 4 with respect to the indices i_e of Step 10. This is formally stated below.

Lemma 6. Let P^A_{α} denote the probability distribution produced in Steps 2 – 11 of Algorithm 4 for a fixed value of α . Then for any request R with covering requirements K(R) = 1,

$$\mathbb{E}_{\pi \sim \mathbf{P}_{\alpha}^{\mathbf{A}}}[\operatorname{AccessCost}(\pi, R)] \leq 2Q \cdot \mathbb{E}[\min_{e \in R} i_{e}] + 1,$$

with i_e as defined in Step 10 of Algorithm 4.

Proof. Let O_i^R denote the set of elements outside R with index value $i_e \leq i$,

$$O_i^R = \{ e \notin R : i_e \le i \}.$$

Notice that Algorithm 4 orders the elements with respect to the values i_e (Step 12). Since the covering requirements of the request R is K(R) = 1,

$$\operatorname{AccessCost}(\pi, R) \le |O_{\min_{e \in R} i_e}^R| + 1.$$

The latter holds since R is covered at the first index in which one of its elements appears (K(R) = 1). As a result,

$$\mathbb{E}_{\pi \sim P_{\alpha}^{A}}[\operatorname{AccessCost}(\pi, R)] \leq \mathbb{E}[|O_{\min_{e \in R} i_{e}}^{R}|] + 1 \leq \sum_{e' \notin R} \Pr[i_{e'} \leq \min_{e \in R} i_{e}] + 1$$

It is not hard to see that,

$$\sum_{e' \notin R} \Pr[i_{e'} \le \min_{e \in R} i_e] + 1 = \mathbb{E}\left[\sum_{e' \notin R} \sum_{j=1}^{\min_{e \in R} i_e} B_{e'j}\right] + 1 \le 2Q \cdot \mathbb{E}[\min_{e \in R} i_e] + 1$$

where the first equality follows by the fact that, once B is fixed, $\Pr[i_e \leq k] = \sum_{j=1}^k B_{ej}$ (Step 10 of Algorithm 4) and the last inequality follows by Case 2 of Lemma 5.

Lemma 7. Let i_R^{α} denote the first position at which $\sum_{j=1}^{i_R^{\alpha}} \sum_{e \in R} A_{ej} \ge \alpha$ then

$$\int_0^1 i_R^\alpha \ d\alpha \le \sum_{i=1}^n \left(1 - \sum_{j=1}^{i-1} \sum_{e \in R} A_{ej} \right)_+ = \mathrm{SW}_R(A)$$

Proof. In order to prove Lemma 7, let us assume that a random variable β is selected according to the uniform probability distribution in [0,1], i.e., with density function $f(\beta) = 1$. As a result, $\int_0^1 i_R^{\alpha} d\alpha = \int_0^1 i_R^{\beta} d\beta = \mathbb{E}[i_R^{\beta}] = \sum_{i=1}^n \Pr[i_R^{\beta} \ge i]$. Since i_R^{β} is the first position at which $\sum_{j=1}^{i_R} \sum_{e \in R} A_{ej} \ge \beta$, $\Pr[i_R^{\beta} \ge i] = \Pr[\beta > \sum_{i=1}^{i-1} \sum_{e \in R} A_{ej}] = \max\left(1 - \sum_{i=1}^{i-1} \sum_{e \in R} A_{ej}, 0\right) \le \sum_{i=1}^n \left(1 - \sum_{i=1}^{i-1} \sum_{e \in R} A_{ej}\right)$

with the second equality following because β is selected according to the uniform distribution in [0, 1]. \Box

To this end we have upper bounded the expected access cost of Algorithm 4 by $\mathbb{E}[\min_{e \in R} i_e]$ (Lemma 4) and lower bounded SW_R(A) by $\int_0^1 i_R^{\alpha} d\alpha$ (Lemma 7). In Lemma 8 we associate these bounds. At this point the role of Condition 1 of Lemma 5 is revealed.

Lemma 8. Let i_R^{α} denote the first position at which $\sum_{j=1}^{i_R^{\alpha}} \sum_{e \in R} A_{ej} \ge \alpha$ then $\mathbb{E}[\min_{e \in R} i_e] \le i_R^{\alpha}/(1-2e^{-\alpha Q}).$ Proof.

$$\begin{aligned} \Pr[\min_{e \in R} i_e \ge 2^k \cdot i_R^{\alpha} + 1] &= & \Pi_{e \in R} \Pr[i_e \ge 2^k \cdot i_R^{\alpha} + 1] \\ &= & \Pi_{e \in R} \Pr[\alpha_e > \sum_{j=1}^{2^k \cdot i_R^{\alpha}} B_{ej}] \\ &= & \Pi_{e \in R} \left(1 - \sum_{j=1}^{2^k \cdot i_R^{\alpha}} B_{ej} \right)_+ \\ &\leq & e^{-\sum_{e \in R} \sum_{j=1}^{2^k \cdot i_R^{\alpha}} B_{ej}} \\ &\leq & e^{-(k+1)Q \sum_{e \in S} \sum_{j=1}^{i_R^{\alpha}} A_{ej}} \\ &\leq & e^{-(k+1)Q\alpha} = p^{k+1} \end{aligned}$$

where the second inequality follows by Case 1 of Lemma $\frac{5}{5}$ and the definition (and manipulation) of matrix B inside Algorithm 4.

$$\mathbb{E}[\min_{e \in R} i_e] = i_R^{\alpha} + \sum_{k=1}^{\infty} \Pr[2^{k-1} \cdot i_R^{\alpha} + 1 \le i_R \le 2^k \cdot i_R^{\alpha}] \cdot 2^k \cdot i_R^{\alpha}$$

$$\le i_R^{\alpha} + \sum_{k=1}^{\infty} 2^k \cdot i_R^{\alpha} \cdot e^{-kQ\alpha} = i_R^{\alpha}/(1 - 2e^{-Q\alpha})$$

Lemma 9. Let $Q := z/\alpha$ for some positive constant z. For any request R with covering requirement K(R) = 1,

$$\mathop{\mathbb{E}}_{\pi \sim \mathcal{P}^A} \left[\operatorname{AccessCost}(\pi, R) \right] \le \left(\frac{4z}{1 - 2e^{-z}} + 1 \right) \cdot \sum_{i=1}^n \left(1 - \sum_{j=1}^{i-1} \sum_{e \in R} A_{ej} \right)_+$$

Proof.

$$\begin{split} \mathbb{E}_{\pi \sim \mathbb{P}^{A}}[\operatorname{AccessCost}(\pi, R)] &= \int_{0}^{1} \mathbb{E}_{\pi \sim \mathbb{P}^{A}}[\operatorname{AccessCost}(\pi, R)] \cdot (2\alpha) \ d\alpha \\ &\leq \int_{0}^{1} 2Q \cdot \mathbb{E}[\min_{e \in R} i_{e}] \cdot (2\alpha) \ d\alpha + \int_{0}^{1} (2\alpha) \ d\alpha \ (Lemma \ 6) \\ &= \int_{0}^{1} 4z \cdot \mathbb{E}[\min_{e \in R} i_{e}] d\alpha + 1 \ (Q = z/\alpha) \\ &\leq \int_{0}^{1} 4z \cdot i_{R}^{\alpha}/(1 - 2e^{-z}) \ d\alpha + 1 \ (Lemma \ 8 \ \text{and} \ Q = z/\alpha) \\ &= \frac{4z}{1 - 2e^{-z}} \int_{0}^{1} i_{R}^{\alpha} \ d\alpha + 1 \ (z = \alpha Q \ \text{is constant}) \\ &\leq \left(\frac{4z}{1 - 2e^{-z}} + 1\right) \sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i-1} \sum_{e \in S} A_{ej}\right)_{+} \ (Lemma \ 8) \\ &= \left(\frac{4z}{1 - 2e^{-z}} + 1\right) \operatorname{SW}_{R}(A) \ (Corollary \ 2) \end{split}$$

Lemma 4 directly follows by setting z := 1.6783 in Lemma 9.

We conclude the section with the following corollary that provides with a quadratic-time algorithm for computing the subgradient in case of *Min-Sum Set Cover problem*.

Corollary 3. Let a doubly stochastic matrix A and a request R. Let i^* denotes the index at which $\sum_{j=1}^{i-1} A_{ej} \leq 1$ and $\sum_{j=1}^{i-1} A_{ej} > 1$. Let also the $n \times n$ matrix B defined as follows,

$$B_{ej} = \begin{cases} i^* - j & \text{if } j \le i^* - 1 \text{ and } e \in R\\ 0 & \text{otherwise} \end{cases}$$

The matrix B (vectorized) is a subgradient of the $SW_R(\cdot)$ at point A.

B.3 Proof of Theorem 2

All steps of Algorithm 2 run in polynomial-time. In Step 3 of Algorithm 2, any $(1 + \alpha)$ -approximation, polynomial-time algorithm for $\min_{R \in [\text{Rem}]^r} \operatorname{AccessCost}(R, A)$ can be used. The first choice that comes in mind is exhaustive search over all the requests of size r, resulting in $\Theta(n^r)$ time complexity. Since the latter is not polynomial, we provide a $(1 + \alpha)$ -approximation algorithm running in polynomial-time in both parameters n and r. For clarity of exposition the algorithm used in Step 3 is presented in Section B.5. In the following we focus on proving Theorem 2.

We remark that by Corollary 2 of Section B.2 and Lemma 3 of Section B.1, for any request R with covering requirement K(R) = 1,

$$\sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i-1} \sum_{e \in R} A_{ej}, 0 \right)_{+} \le (1+\epsilon) \cdot \text{FAC}_{R}(A) \text{ for any } A \in \text{DS}$$

where ϵ is the parameter used in Definition 2. As a result, Theorem 2 follows directly by Theorem 5, which is stated below and proved in the next section.

Theorem 5. Let $\pi_A \in [n!]$ denote the permutation of elements produced by Algorithm 2 when the doubly stochastic matrix $A \in DS$ is given as input. Then for any request R with $|R| \leq r$ and K(R) = 1,

AccessCost
$$(\pi_A, R) \le 2(1+\alpha)^2 r \cdot \sum_{i=1}^n \left(1 - \sum_{j=1}^{i-1} \sum_{e \in R} A_{ej}\right)_+.$$

B.4 Proof of Theorem 5

Consider a request $R \in [n^r]$ such that

$$(L-1) \cdot r + 1 \le \operatorname{AccessCost}(\pi_A, R) \le L \cdot r \tag{5}$$

for some integer L. Since K(R) = 1 this means that the first element of R appears between positions $(L-1) \cdot r + 1$ and $L \cdot r$ in permutation π_A .

To simplify notation we set $\operatorname{Cost}(A, R) := \sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i-1} \sum_{e \in R} A_{ej} \right)_{+}$. To prove Theorem 5 we show the following, which can be plugged in (5) and give the result:

$$\operatorname{Cost}(A, R) \ge \frac{L}{2(1+\alpha)^2}$$

Let R_{ℓ} denote the request of size r composed by the elements lying from position $(\ell - 1) \cdot r + 1$ to $\ell \cdot r$ in the produced permutation π_A . Recall the minimization problem of Step 3. R_{ℓ} is a $(1+\alpha)$ approximately optimal solution for that problem and thus its corresponding cost is at most $(1+\alpha)$ times the corresponding cost of any other same-cardinality subset of the remaining elements. Since in π_A all the elements of R lie on the right of position $(L-1) \cdot r$, all elements of R are present at the L-th iteration and thus,

$$\operatorname{Cost}(A, R_L) \le (1 + \alpha) \cdot \operatorname{Cost}(A, R)$$

Moreover, by the same reasoning,

$$\operatorname{Cost}(A, R_{\ell}) \leq (1 + \alpha) \cdot \operatorname{Cost}(A, R_L), \text{ for all } \ell = 1, \dots, L.$$

Thus it suffices to show that $Cost(A, R_L) \ge L/2(1 + \alpha)$. The latter is established in Lemma 10, which concludes the section.

Lemma 10. Let R_1, R_2, \ldots, R_L be disjoint requests of size r such that for all $\ell = 1, \ldots, L$, $Cost(A, R_\ell) \leq (1 + \alpha) \cdot Cost(A, R_L)$. Then,

$$\operatorname{Cost}(A, R_L) \ge \frac{L}{2(1+\alpha)}$$

Proof. For each request R_{ℓ} we define the quantity $B_{\ell i}$ as follows:

$$B_{\ell i} = \begin{cases} \sum_{e \in R_{\ell}} A_{ei} & \text{if } \sum_{j=1}^{i} \sum_{e \in R_{\ell}} A_{ej} < 1\\ 1 - \sum_{j=1}^{i-1} \sum_{e \in R_{\ell}} A_{ej} & \text{if } \sum_{j=1}^{i} \sum_{e \in R_{\ell}} A_{ej} \ge 1 \text{ and } \sum_{j=1}^{i-1} \sum_{e \in R_{\ell}} A_{ej} < 1\\ 0 & \text{otherwise} \end{cases}$$

Observation 1. The following 3 equations hold,

- 1. $\sum_{i=1}^{n} B_{\ell i} = 1.$
- 2. $B_{\ell i} \leq \sum_{e \in R_{\ell}} A_{ei}$.

3.
$$\operatorname{Cost}(A, R_{\ell}) = \sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i-1} \sum_{e \in R_{\ell}} A_{ej} \right)_{+} = \sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i-1} B_{\ell j} \right)_{+}$$

Since $(1 + \alpha) \cdot \operatorname{Cost}(A, R_L) \ge \operatorname{Cost}(A, R_\ell)$ for all $\ell = 1, \dots, L$,

$$Cost(A, R_L) \geq \frac{1}{1+\alpha} \cdot \frac{1}{L} \sum_{\ell=1}^{L} Cost(A, R_\ell) = \frac{1}{1+\alpha} \cdot \left[\frac{1}{L} \sum_{\ell=1}^{L} \sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i-1} B_{\ell j} \right) \right]$$
$$= \frac{1}{1+\alpha} \cdot \left[n - \frac{1}{L} \sum_{\ell=1}^{L} \sum_{i=1}^{n} \sum_{j=1}^{i-1} B_{\ell j} \right]$$
$$= \frac{1}{1+\alpha} \cdot \left[n - \frac{1}{L} \sum_{i=1}^{n} \sum_{j=1}^{i-1} C_j \right] \quad (\text{where } C_j = \sum_{\ell=1}^{L} B_{\ell j})$$
$$= \frac{1}{1+\alpha} \cdot \left[n - \frac{1}{L} \sum_{i=1}^{n} (n-i) \cdot C_i \right] = \frac{1}{1+\alpha} \cdot \left[n - \frac{n}{L} \sum_{i=1}^{n} C_i + \frac{1}{L} \sum_{i=1}^{n} i \cdot C_i \right]$$

Observe that $\sum_{i=1}^{n} C_i = \sum_{i=1}^{n} \sum_{\ell=1}^{L} B_{\ell i} = \sum_{\ell=1}^{L} \sum_{i=1}^{n} B_{\ell i} = L$, where in the last equality we used $\sum_{i=1}^{n} B_{\ell i} = 1$. Thus we get that

$$\operatorname{Cost}(A, R_L) \ge \frac{1}{1+\alpha} \left[\frac{1}{L} \sum_{i=1}^n i \cdot C_i \right]$$

To this end, to conclude the result, one can prove that $\sum_{i=1}^{n} i \cdot C_i \ge L^2/2$ using that $\sum_{i=1}^{n} C_i = L$ and $C_i \le 1$. $C_i \le 1$ follows by the *disjoint property* of the requests R_1, \ldots, R_L . More precisely,

$$C_{i} = \sum_{\ell=1}^{L} B_{\ell i} \leq \sum_{\ell=1}^{L} \sum_{r \in R_{\ell}} A_{r i}$$
$$\leq \sum_{e=1}^{n} A_{e i} = 1$$

where the first inequality follows from Observation 1 and the last inequality by R_1, \ldots, R_L not sharing any element.

B.5 Implementing Step 3 of Algorithm 2 in Polynomial-Time

In this section we present a polynomial time algorithm implementing Step 3 of Algorithm 2. More precisely, we present a *Fully Polynomial-Time Approximation Scheme (FTPAS)* for the combinatorial optimization problem defined below, in Problem 1.

Problem 1. Given an $n \times n$ doubly stochastic matrix A and a set of elements $Rem \subseteq \{1, \ldots, n\}$. Select the r elements of $Rem \ (R^* \subseteq Rem \ with \ R^* = r)$ minimizing,

$$\sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i-1} \sum_{e \in R^*} A_{ej} \right)_{+}.$$

In fact we present a $(1 + \alpha)$ -approximation algorithm for a slightly more general problem, Problem 2.

Problem 2. Given a set of m vectors B_1, \ldots, B_m , of size n such that,

$$0 = B_{e1} \le B_{e2} \le \ldots \le B_{en} = 1$$
, for each $e = 1, \ldots, m$

Select the r vectors $(R^* \subseteq [m] \text{ with } R^* = r)$ minimizing

$$\sum_{i=1}^{n} \left(1 - \sum_{e \in R^*} B_{ei} \right)_+$$

Setting $B_{ei} = \sum_{j=1}^{i-1} A_{ej}$, one can get Problem 1 as a special case of Problem 2.

Theorem 6. There exists a $(1 + \alpha)$ -approximation algorithm for Problem 2 that runs in $\Theta(n^4r^3/\alpha^2)$ steps.

The $(1 + \alpha)$ -approximation algorithm of Problem 2 heavily relies on solving the Integer Linear Program defined in Problem 3.

Problem 3. Given a set of m triples of integers (w_e, c_e, d_e) such that $c_e, d_e \ge 0$ for each $e \in \{1, m\}$ and two positive integers C, D,

$$\begin{array}{ll} \mbox{minimize} & \sum_{e=1}^{m} w_e x_e \\ \mbox{subject to} & \sum_{e=1}^{m} c_e x_e \geq C \\ & \sum_{e=1}^{m} d_e x_e \leq D \\ & \sum_{e=1}^{m} x_e = r \\ & x_e \in \{0,1\} \quad e = 1, ..., m \end{array}$$

Lemma 11. Problem 3 can be solved in $\Theta(n \cdot C \cdot D \cdot r)$ steps via Dynamic Programming.

Proof. Let DP(n, r, C, D) denotes the value of the optimal solution. Then

$$DP(n, r, C, D) = \min (DP(n - 1, r - 1, C - x_n, D - d_n), DP(n - 1, r, C, D))$$

In the rest of the section, we present the $(1 + \alpha)$ -approximation algorithm for Problem 2 as stated in Lemma 6 using the algorithmic primitive of Lemma 11.

We first assume the entries of the input vectors are multiples of small constant $\alpha \ll 1$, $B_{ei} = k_{ei} \cdot \alpha$ for some integer k_{ei} . Under this assumption we can use the algorithm (stated in Lemma 11) for Problem 3 to find the *exact* optimal solution of Problem 2 in $\Theta(n^2 r/\alpha^2)$ steps. More precisely, for a fixed index k, let OPT_k denotes the optimal solution among the set of vectors of size r that additionally satisfy,

$$\sum_{e \in R} B_{e(k-1)} < 1 \text{ and } \sum_{e \in R} B_{ek} \ge 1$$
(6)

It is immediate that $OPT = \arg\min_{1 \le k \le n} OPT_k$ and thus the problem of computing OPT reduces into computing OPT_k for each index k. We can efficiently compute OPT_k for each index k by solving an appropriate instance of Problem 3. To do so, observe that for any set of vectors R satisfying the constraints of Equation (6) for the index k,

$$\sum_{i=1}^{n} \left(1 - \sum_{e \in R} B_{ei} \right)_{+} = \sum_{i=1}^{k-1} \left(1 - \sum_{e \in R} B_{ei} \right) = \sum_{e \in R} \underbrace{\sum_{i=1}^{k-1} \left(\frac{1}{r} - B_{ei} \right)}_{w_{e}}$$

where the first equality comes from the fact that $B_{e1} \leq \ldots \leq B_{en}$. It is not hard to see that OPT_k can be computed via solving the instance of Problem 3 with triples $\left(w_e = \sum_{i=1}^{k-1} \left(\frac{1}{r} - B_{ei}\right), c_e = B_{e(k-1)}, d_e = B_{ek}\right)$ for each $e = 1 \ldots, m, D = 1$ and C = 1. Moreover by Lemma 11 this is done in $\Theta(nr/\alpha^2)$ steps. Thus the overall time complexity in order to compute the optimal solution of Problem 2 (in case the entries B_{ei} are multiplies of α) is $\Theta(n^2r/\alpha^2)$.

We now remove the assumption that the entries B_{ei} are multiples of α via relaxing the optimality guarantees by a factor of $(1 + \alpha)$. We first construct a new set of vector with entries *rounded* to the closest multiple of α , $\hat{B}_{ei} = \lfloor B_{ei}/\alpha \rfloor \cdot \alpha$ and solve the problem as if the entries where \hat{B}_{ei} in $\Theta(n^2r/\alpha^2)$ steps. The quality of the produced solution, call it Sol can be bounded as follows

$$\sum_{i=1}^{n} \left(1 - \sum_{e \in \text{Sol}} B_{ei} \right)_{+} \leq \sum_{i=1}^{n} \left(1 - \sum_{e \in \text{Sol}} \hat{B}_{ei} \right)_{+}$$
$$\leq \sum_{i=1}^{n} \left(1 - \sum_{e \in \text{Sol}} \hat{B}_{ei} \right)_{+}$$
$$\leq \sum_{i=1}^{n} \left(1 - \sum_{e \in \text{OPT}} (B_{ei} - \alpha) \right)_{+}$$
$$\leq \sum_{i=1}^{n} \left(1 - \sum_{e \in \text{OPT}} B_{ei} \right)_{+} + nr \cdot \alpha$$

Setting $\alpha := \alpha'/nr$, we get that

$$\sum_{i=1}^{n} \left(1 - \sum_{e \in \text{Sol}} B_{ei} \right)_{+} \leq \sum_{i=1}^{n} \left(1 - \sum_{e \in \text{OPT}} B_{ei} \right)_{+} + \alpha' \leq (1 + \alpha') \sum_{i=1}^{n} \left(1 - \sum_{e \in \text{OPT}} B_{ei} \right)_{+}$$

since $B_{e1} = 0$ for all e. Thus, the overall time needed to produce a $(1 + \alpha')$ -approximate solution is $\Theta(n^4 r^3/(\alpha')^2)$, proving the result.

B.6 A simple heuristic for Problem 1

In this section we present a simple heuristic for Problem 1 that can be a good alternative of the algorithm elaborated in Section B.5. We remark that Algorithm 5 may provide highly sub-optimal solutions in the worst case however our experiments suggest that it works well enough in practice. As explained in Section 5, in our experimental evaluations we use this heuristic to implement Step 3 of Algorithm 2. This was done since this heuristic is easier and faster to implement.

Algorithm 5 A simple heuristic for Problem 1

Input: A doubly stochastic matrix $A \in DS$. **Output:** A set R (of r elements) approximating Problem 1

1: $R = \emptyset$ 2: Target = (1, ..., 1)3: for $\ell = 1$ to r do 4: $e_{\ell} \leftarrow \arg\min_{e \in \{1,...,n\}/R} \left(\sum_{j=1}^{n} \max(\operatorname{Target}[j] - \sum_{s=1}^{j-1} A_{ej}, 0) \right)$ 5: $R \leftarrow R \cup \{e_{\ell}\}$ 6: Target $\leftarrow (\operatorname{Target} - (A_{e1}, ..., A_{en}))_{+}$ 7: end for 8: Output the set of elements R.