Constant Inapproximability for Fisher Markets

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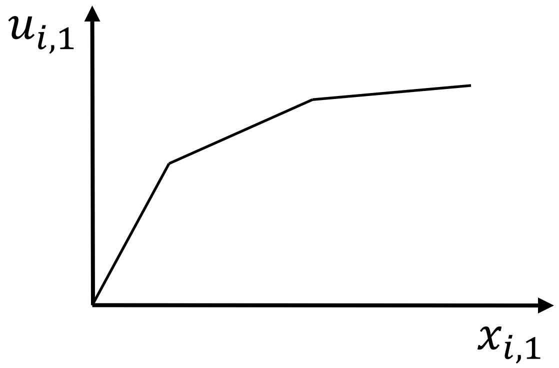
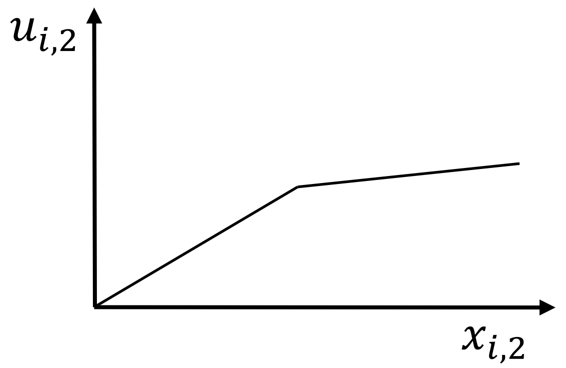
Fisher Markets

- Set of divisible goods G, one unit of each good
- Set of buyers B
- Buyer $i \in B$ has budget e_i
- Each buyer i has a utility function $u_i: R_{\geq 0}^{|G|} \to R_{\geq 0}$
- Under allocation $x_i \in R_{\geq 0}^{|G|}$ for buyer i, where $x_{i,j} \geq 0$ is the allocation of good j, $u_i(x_i)$ denotes the utility of the buyer

SPLC utilities

- Additive **S**eparable **P**iecewise **L**inear **C**oncave
- $u_i(x_i) = \sum_{j \in G} u_{i,j}(x_{i,j})$ where $u_{i,j}: R_{\geq 0} \to R_{\geq 0}$ satisfies
  - $u_{i,j}(0) = 0$
  - $u_{i,j}$ is continuous and piecewise-linear
  - $u_{i,j}$ is concave but non-decreasing

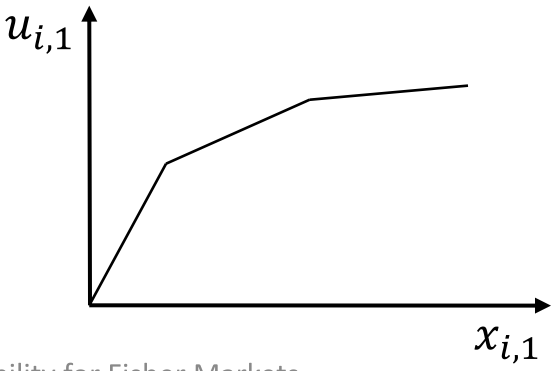


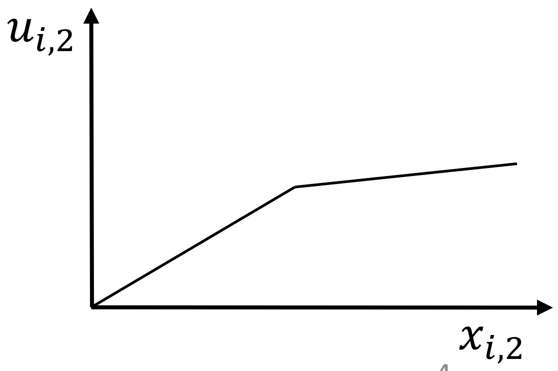


Fisher Markets

- Set of divisible goods G, one unit of each good
- Set of buyers B
- Buyer $i \in B$ has budget e_i
- Each buyer i has an SPLC utility function $u_i: R_{\geq 0}^{|G|} \rightarrow R_{\geq 0}$
- Given price vector $p \in R_{\geq 0}^{|G|}$,
  $OPT_i(p) \subseteq R_{\geq 0}^{|G|}$ is the set of optimal bundles for buyer i

$$\begin{aligned} \max \quad & u_i(x_i) \\ \text{s.t.} \quad & \sum_{j \in G} p_j x_{i,j} \leq e_i \\ & x_{i,j} \geq 0 \quad \forall j \in G. \end{aligned}$$





Competitive Equilibria

- For every $\varepsilon \geq 0$, an ε - approximate market equilibrium is a price vector p and allocation vector $x = (x_i)_{i \in B}$ s.t.
  1. Every buyer buys an optimal bundle, i.e., $x_i \in OPT_i(p)$
  2. For every good j, the market approximately clears up to ε units, i.e., $\left|\sum_{i \in B} x_{i,j} - 1\right| \leq \varepsilon$

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Sufficient Condition: For every buyer i there is a good j s.t. $u_{i,j}$ is a strictly increasing function, i.e. the buyer is not satiated

Every Fisher market that satisfies the sufficient condition possesses at least one market equilibrium

Complexity of Market Equilibria

Every Fisher market that satisfies the sufficient condition possesses at least one ε - approximate market equilibrium

Input: A Fisher market with SPLC utility functions that satisfy the sufficient condition.

Task: Compute an ε - approximate market equilibrium

Complexity of Market Equilibria

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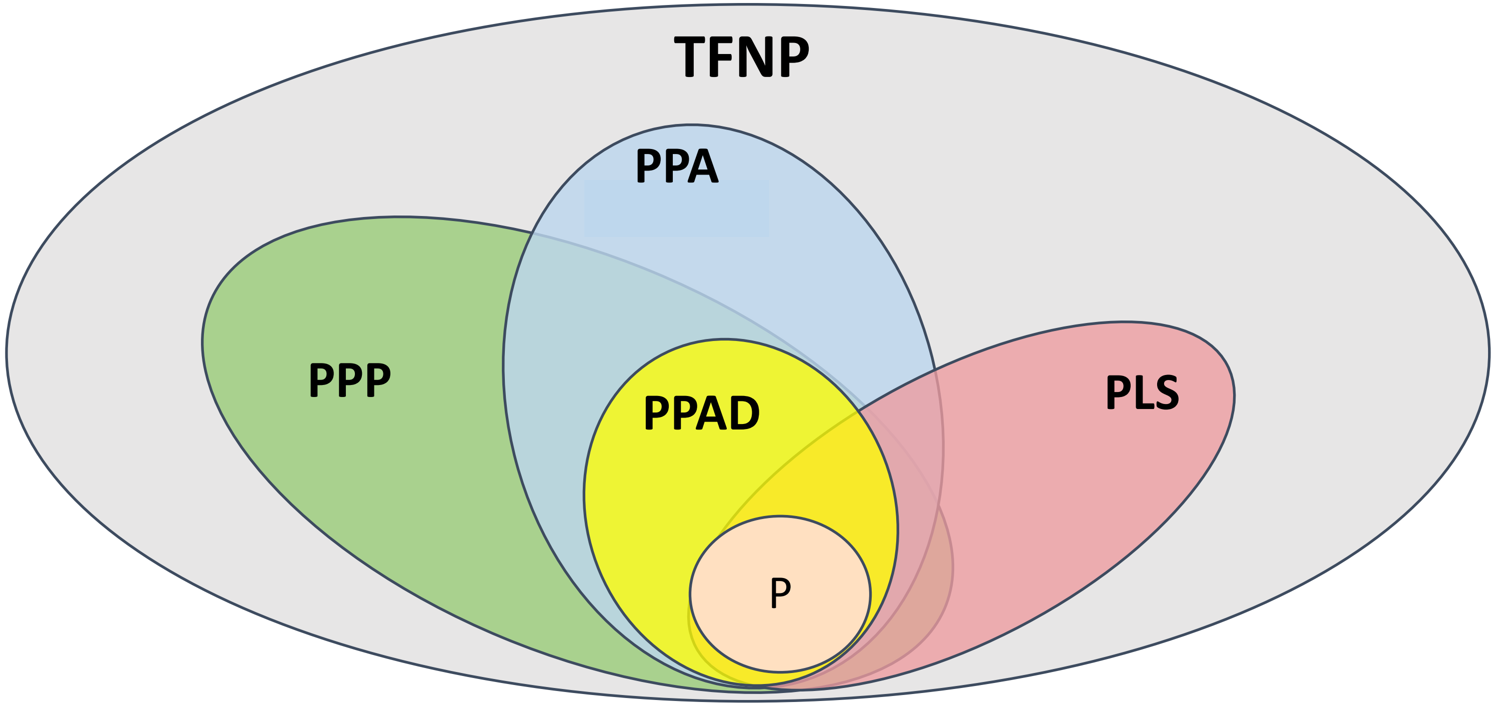


NP Total Search (TFNP) problem!
- Total: there is always a solution
- NP: it is easy to verify solutions

Can a TFNP problem be NP-hard?

Not unless co-NP = NP…

Related work

- PPAD-hardness for inverse polynomial ε

  (Vazirani and Yannakakis, Chen and Teng)

- Polynomial time algorithms
  - Linear Utilities (Devanur et al, Orlin, Vegh)
  - Homogeneous (Eisenberg)
  - Weak gross substitutes (Codenotti et al)
  - Constant number of agents or goods (Devanur and Kannan)
  - Fixed parameter approximation scheme wrt buyers (Garg et al)

More Related work

- **Matching Markets**
  - Constant number of buyers or goods (Alaei et al)
  - Dichotomous utilities (Vazirani and Yannakakis)
  - Hylland-Zeckhauser markets (Hylland and Zeckhauser, Braverman, Chen et al.)

- **Fisher markets with constraints**
  - Utilities depend on spending constraints (Birnbaum et al., Devanur, Vazirani)
  - Linear constraints (Jalota et al.)

Even More Related work

- Arrow-Debreu exchange
  - PPAD-hardness:
    1/poly (Chen et al)
    constant, yet uspecified, ε (Rubinstein)
  - Polynomial time algorithms
    Linear utilities (Duan and Melhorn, Duan et al., Garg and Vegh, Jain, Ye)
    Weak gross substitutes (Bei et al, Codenotti et al., Garg et al.)

Our results

It is PPAD-complete to compute an ε - approximate market equilibrium in Fisher markets with SPLC utilities, for any constant $\varepsilon < 1/11$.

It is PPAD-complete to compute an ε - approximate market equilibrium in Arrow-Debreu exchange markets with SPLC utilities, for any constant $\varepsilon < 1/11$.

Reduction from Pure-Circuit problem

The Pure-Circuit Problem

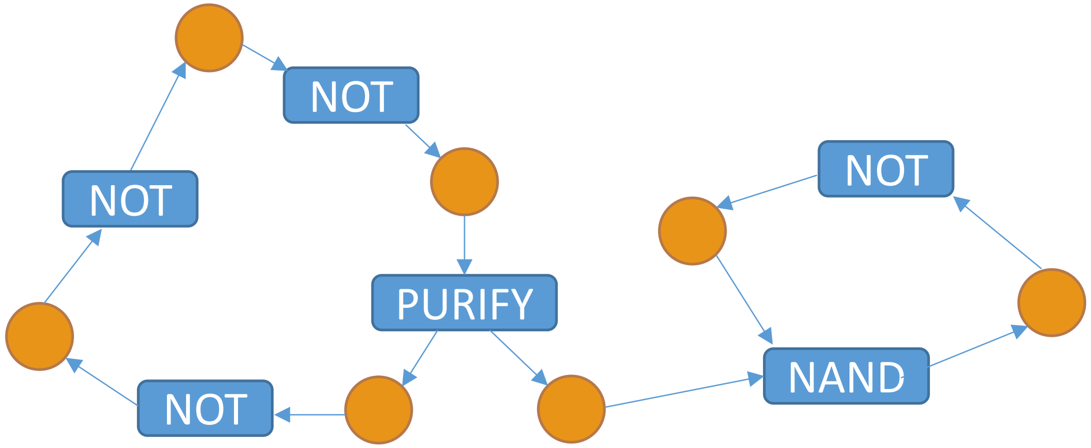
Input: A Boolean circuit where

1. The circuit can have cycles
2. Variables take values in $\{0, 1, \bot\}$ instead of just $\{0, 1\}$
3. In addition to the standard logical gates (NOT, OR, AND), the circuit can also have "PURIFY" gates

Goal: Assign a value (in $\{0, 1, \bot\}$), such that all gates are "satisfied"

The Pure-Circuit Problem

Goal: Assign a value (in $\{0, 1, \bot\}$), such that all gates are "satisfied"



Theorem (DFHM):
Pure-Circuit is PPAD-complete even if the circuit has gates in {NOT, NAND, PURIFY}

Goal: Assign a value (in $\{0,1,\perp\}$), such that all gates are "satisfied"

u	v
0	1
1	0
$\perp$	$\{0,1,\perp\}$

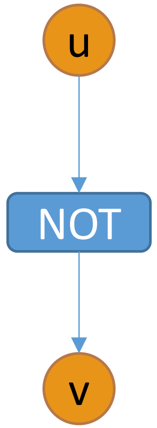
NOT gate

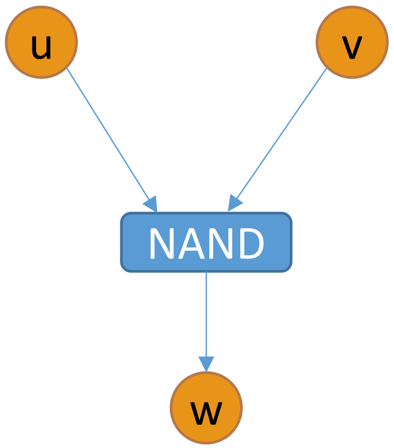
u	v	w
1	1	0
0	$\{0,1,\perp\}$	1
$\{0,1,\perp\}$	0	1
Else		$\{0,1,\perp\}$

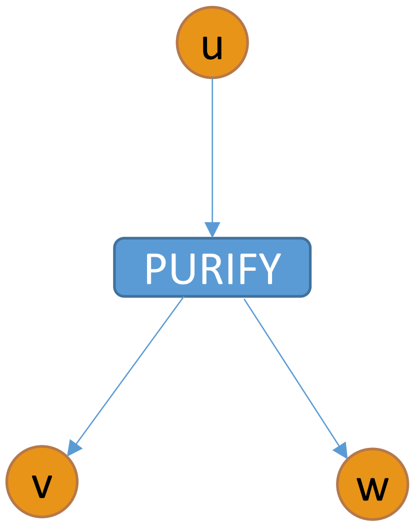
NAND gate

u	v	w
0	0	0
1	1	1
$\perp$	At least one output in $\{0,1\}$	

PURIFY gate







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Reduction from Pure-Circuit problem

High-level idea of our reduction

Given: An instance of Pure-Circuit

Goal: Construct a Fisher market

Idea: Create one good for each variable in the Pure-Circuit instance plus a "reference" good.

Buyers (and auxiliary buyers) will help us to implement the gates

Interpretation:

- If a good has "low" price → variable value = 0
- If a good has "high" price → variable value = 1
- Otherwise → variable value = ⊥

Overview: Reference good ref

- In every equilibrium it has price p_{ref} close to 1
- We ensure this via a reference buyer

$$u_{b_{\mathrm{ref}},j}(x) = \begin{cases} x & \text{if } j = \mathsf{ref}, \\ 0 & \text{otherwise.} \end{cases}$$

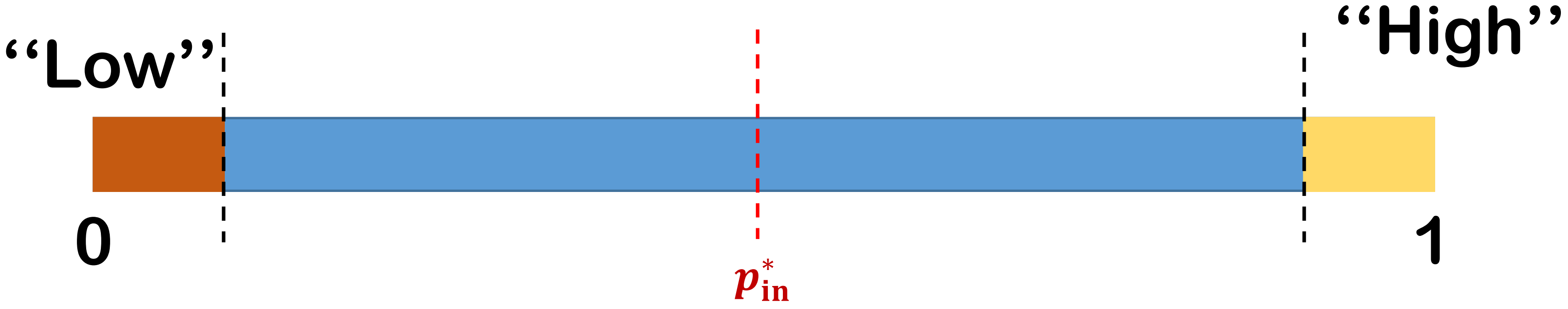
- Every other buyer wants the reference good, but we ensure that the demand from them is significantly smaller than 1

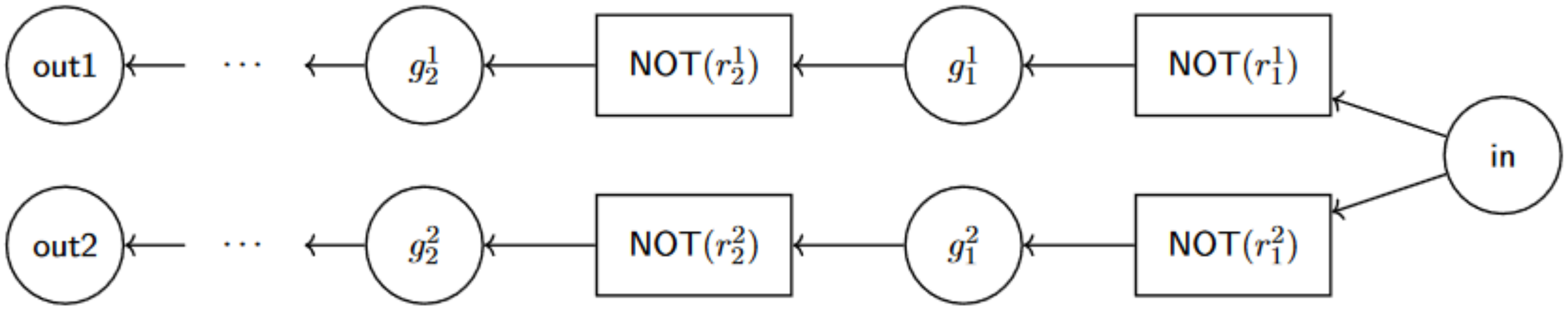
Overview: Variable encodings

$s < 1/\text{poly}, \;\; a$ ‘large’

- ➢ In every equilibrium it has price p_{ref} close to 1

Overview: PURIFY gates





- If $p_{\text{in}} \geq p_{\text{in}}^*$, then $p_{\text{out1}} = H$
- If $p_{\text{in}} \leq p_{\text{in}}^*$, then $p_{\text{out2}} \leq L$

Conclusions

- First constant inapproximability for Fisher markets
- Use Pure-Circuit to prove constant inapproximability for Hylland-Zeckhauser?
- Can we improve ε?
- Upper bounds?

Thank you!

Questions?