When Does Diversity of Agent Preferences Improve Outcomes in Selfish Routing?

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Abstract

We seek to understand when heterogeneity in agent preferences yields improved outcomes in terms of overall cost. That this might be hoped for is based on the common belief that diversity is advantageous in many multi-agent settings. We investigate this in the context of routing. Our main result is a sharp characterization of the network settings in which diversity always helps, versus those in which it is sometimes harmful.

Specifically, we consider routing games, where diversity arises in the way that agents trade-off two criteria (such as time and money, or, in the case of stochastic delays, expectation and variance of delay). Our main contributions are: 1) A participant-oriented measure of cost in the presence of agent diversity; 2) A full characterization of those network topologies for which diversity always helps, for all latency functions and demands.

1 Introduction

It is a common belief that diversity helps. In non-cooperative multi-agent systems, where a central theme is the tension between selfish behavior and social optimality—can diversity of agent preferences help to bring us closer to the coveted social optimality? We provide an answer to this question in the context of non-atomic selfish routing, where diversity naturally arises in the way agents trade-off two criteria, for example, time and cost, or, in the presence of uncertain delays, expectation and variance of delay.

As we shall see, there is no unique answer. Rather, it depends on the setting. To address our question we need to specify how to measure the cost of an outcome, and define a benchmark setting with no agent heterogeneity. To measure the cost of an outcome, we treat an agent's cost as the sum of two terms associated with two criteria: If we let ℓ_P denote the cost of one criterion (e.g., the latency) over a path P, and v_P be the cost of the second criterion, then the overall cost is given by $\ell_P + r \cdot v_P$, where r is our diversity parameter. The special case of $r = 0$ corresponds to indifference to the second criterion and results in the classic selfish routing model where agents simply minimize travel time.

A first approach to measuring the effect of diversity might be to compare the cost of an outcome with $r = 0$ (i.e., just the total latency) to that with other values of r, including possibly mixed values of r across the population being routed. However, this approach does not pinpoint the gains and losses from agent heterogeneity as opposed to agent homogeneity; rather, it (mostly) pinpoints the gains and losses depending on whether players are affected by the second criterion or not. Instead, we focus on the sum of the costs incurred by the agents as measured by their cost functions, and compare costs incurred by a heterogeneous population of agents to those incurred by an equivalent population of homogeneous agents. What are equivalent populations? Suppose the heterogeneous population's diversity profile is given by a population density function $f(r)$. Then, we define the corresponding homogeneous population to have the single diversity parameter $\bar{r} = \int r f(r) dr$. In addition, we require the two populations to have the same size, in the sense that the total source-to-sink flows that they induce are equal.

Contribution. We fully characterize the topology of networks for which diversity is never harmful, regardless of the demand size and the distribution of the diversity parameter (discrete or continuous). We do so both for single and multi-commodity networks.

For single-commodity networks it turns out that this topology is that of series-parallel networks. In Theorem 1, we show that if the network is series-parallel, then diversity only helps for any choice of demand and edge functions. The key observation is that there is a path for which the homogeneous flow is at least as large as the heterogeneous flow. As the cost of the homogeneous flow is the same on all used paths, while the cost of each unit of heterogeneous flow is lowest on the path it uses, one can then deduce that the cost of the heterogeneous flow is at most that of the homogeneous flow. To show necessity, we first provide an instance on the Braess graph for which diversity is harmful, and then show how to embed it in any non-series-parallel graph.

In multi-commodity networks, by the result above, each commodity must route its flow through a series-parallel subnetwork. But, as Proposition 2 shows, this is not enough, and the way in which these series-parallel networks overlap needs to be constrained. The necessary constraint is exactly captured by the class of *block-matching* networks, defined in this paper. Sufficiency in this case then follows quite easily

from the same result for the single commodity case.

The main technical challenge is to show necessity. To this end, assuming diversity does no harm, we show, via a case analysis, how the subnetworks of the commodities may overlap. First, in Proposition 2, we give an instance on a network of two commodities and three paths for which diversity hurts. Then we mimic this instance on a general network. The difficult part is to choose the corresponding paths for the mimicking, so that, in the created instance, all the flow under both equilibria goes through these paths. The challenge is that the commodities' subnetworks may overlap in subtle ways.

Related work. Our work was inspired by the selfish routing literature where agents have to tradeoff two criteria such as time versus tolls, or expected travel time versus variance. In the former setting, early results (e.g., [Beckmann *et al.*, 1956]) showed that tolls can help implement the social optimum as an equilibrium, when agents all have the same linear objective function combining time and money. Much more recently, these results were extended to the case where agents trade-off travel time and money differently, by Cole *et al.* (2003) and Fleischer (2005) for the single commodity case, and by Karakostas and Kolliopoulos (2004a) and Fleischer *et al.* (2004) for the multicommodity case. We remark that the above works, apart from [Cole *et al.*, 2003], consider only the case where the social welfare is defined as the total travel time, whereas in our work we consider the total agent cost, which encapsulates both criteria. This, for example, is also the case for Christodoulou *et al.* (2014) and Karakostas and Kolliopoulos (2004b). In stochastic selfish routing and related frameworks, most models assume homogeneous agents (e.g., [Piliouras *et al.*, 2013], [Nikolova and Stier-Moses, 2015], [Lianeas *et al.*, 2016], [Kleer and Schäfer, 2016]) and study the degradation of a network's performance due to risk aversion. Fotakis *et al.* (2015) considered games with heterogeneous risk-averse players and showed how uncertainty can be used to improve a network's performance.

Characterizing the topology of networks that satisfy some property is a common theme in computer science. Relevant to our work, Epstein *et al.* (2009) characterized the topology of single-commodity networks for which all Nash equilibria are social optima (under bottleneck costs), and Milchtaich (2006) characterized the topology of single-commodity networks which do not suffer from the Braess Paradox for any cost functions. Chen *et al.* (2015) fully characterized the topology of multi-commodity networks that do not suffer from the Braess paradox. These characterizations appear similar to ours, although there does not seem to be any other connection between the two problems, as (i) there are instances where diversity helps while the Braess Paradox occurs and others where diversity hurts but the paradox does not occur, and (ii) the Braess Paradox may occur in series-parallel networks when considering selfish routing with heterogeneity in agent preferences, which is not the case for the classic selfish routing model.

In the following works, the characterizing topology for the corresponding question (for a single commodity) is similar to ours. Fotakis and Spirakis (2008) considered atomic games and proved that series-parallel networks are the largest class of networks for which strongly optimal tolls are known to

exist. Nikolova and Stier-Moses (2015) considered homogeneous agents and a social cost function that does not account for the second criterion; They showed that series-parallel networks admit the best bound on the degradation of the network due to risk aversion. Theorem 4 of Acemoglu *et al.* (2016), proves that series-parallel networks are the characterizing topology for what they call the Informational Braess Paradox with Restricted Information Sets; this theorem compares the cost of one agent type before and after more information is revealed to agents of that type, but does not consider the change in the cost of other agent types. In contrast, our work considers non-atomic games with heterogeneous agents and bounds the overall costs faced by the collection of agents. Most relevant to our work is [Meir and Parkes, 2014] and its Theorem 3.1 as it implies that for series-parallel networks the cost of an agent of average parameter only increases when switching from the heterogeneous instance to the corresponding homogeneous one and thus for our sufficiency theorems, one is left to prove that the heterogeneous equilibrium cost is no greater than the cost of an agent of average parameter (though we give a different proof).

2 Preliminaries

Consider a directed multi-commodity network $G = (V, E)$ with an aggregate demand of d_k units of flow between origin-destination pairs (s_k, t_k) for $k \in K$. We let $\mathcal{P}_k$ be the set of all paths between s_k and t_k, and $\mathcal{P} := \cup_{k \in K} \mathcal{P}_k$ be the set of all origin-destination paths. We let $[m]$ denote $\{1, \ldots, m\}$. We assume that $K = [m]$, for some m. The agents in the network—i.e., the players of the game—must choose routes that connect their origins to their destinations. We encode the collective decisions of agents in a flow vector $f = (f_\pi)_{\pi \in \mathcal{P}} \in \mathbb{R}_+^{|\mathcal{P}|}$ over all paths. Such a flow is feasible when demands are satisfied, as given by constraints $\sum_{\pi \in \mathcal{P}_k} f_\pi = d_k$ for all $k \in K$. For simplicity, we let f_e denote the flow on edge e; note that $f_e = \sum_{\pi: e \in \pi} f_\pi$. When we need multiple flow variables, we use the analogous notation g, g_π, g_e.

The network is subject to congestion that affects two criteria the players consider. These two criteria are modeled by two *edge-dependent functions* that take as input the edge flow f_e of e, for each edge e: a latency function $\ell_e(x)$ assumed to be continuous and non-decreasing, and a deviation function $\sigma_e(x)$ assumed to be continuous (but not necessarily non-decreasing).

Throughout the paper we refer to the agent's objective as the cost along a route. Formally, for a given agent, on letting $\ell_\pi(f) = \sum_{e \in \pi} \ell_e(f_e)$ and $\sigma_\pi(f) = \sum_{e \in \pi} \sigma_e(f_e)$, for a constant $r \geq 0$ that quantifies the agent diversity parameter, the agent's cost along route π under flow f is

$$c_\pi^r(f) = \sum_{e \in \pi} \ell_e(f_e) + r \sum_{e \in \pi} \sigma_e(f_e) = \ell_\pi(f) + r\sigma_\pi(f) \quad (1)$$

We assume that for any edge and any agent's diversity parameter r, the functions ℓ_e and $\ell_e + r\sigma_e$ are non-decreasing. Note that if there is an upper bound $r_{\max}$ on the possible values of the diversity parameter r, then the latter assumptions do not require σ_e to be non-decreasing. This is desirable because, for example, in risk-averse selfish routing where σ_e models the variance, σ_e may be decreasing in the flow.

Players Heterogeneity. We assume that there may be more than one value of the diversity parameters r for the players routing commodity k. We use the term *single-minded* to refer to players with $r = 0$. We handle both the cases of a continuous and a discrete distribution of the diversity parameter among the players, though, for our results, we need only consider the discrete case.[1]

For a discrete distribution of, say, n discrete values $r_1^k, \ldots, r_n^k$, the demand d_k is a vector $d_k = (d_1^k, \ldots, d_n^k)$ where each d_i^k denotes the total demand of Commodity k with diversity parameter r_i^k. We let d^k denote Commodity k's total demand, $d^k = \sum_{i=1}^n d_i^k$. Variables f_π^r and f_e^r denote the flow of diversity parameter r on path π and edge e, respectively. Formally, an instance is described by the tuple $(G, \{(\ell_e, \sigma_e)\}_{e \in E}, \{(s_k, t_k)\}_{k \in K}, \{d_k\}_{k \in K}, \{r_k\}_{k \in K})$, where $r_k = (r_1^k, \ldots, r_n^k)$ is the vector of different diversity parameters encountered in the heterogeneous population.

Equilibrium flows. The Wardrop equilibrium of an instance is a flow f such that for every $k \in K$, for every path $\pi \in \mathcal{P}_k$ with positive flow, and any diversity parameter r on it, the path cost $c_\pi^r(f) \leq c_{\pi'}^r(f)$ for all paths $\pi' \in \mathcal{P}_k$.

From here on, we shall refer to the Wardrop equilibrium as the equilibrium. Our goal is to compare the total agent cost at the equilibrium of an instance with a population that has heterogeneous diversity parameters, to the total agent cost at the equilibrium of the same instance but with the population of each commodity keeping its magnitude but changed to be homogeneous, with diversity parameter equal to the expected value of the diversity parameter distribution in the heterogeneous population of the commodity. To differentiate more easily, for a heterogeneous instance we call the former the *heterogeneous equilibrium* and the latter the (corresponding) *homogeneous equilibrium*. We usually denote the heterogeneous equilibrium by g and the homogeneous equilibrium by f. The existence of both equilibria is guaranteed by e.g. [Schmeidler, 1973, Theorem 2]. We note here that in general we do not need uniqueness of equilibria, neither for the edge costs nor for the edge flows. Our results hold for any arbitrary pair of heterogeneous and homogeneous equilibria of the corresponding instances. Also, as for classic routing games, without loss of generality (WLOG) we may assume that equilibrium flows are acyclic.

Total Costs. For a heterogeneous equilibrium flow vector g, the *heterogeneous total cost* of Commodity k is denoted by $C^{k,ht}(g) = \sum_{j=1\ldots n} d_j^k c^{k,r_j^k}(g)$ where $c^{k,r_j^k}(g)$ denotes the common cost at equilibrium g for players of diversity parameter r_j^k in Commodity k. The *heterogeneous total cost* of g is then $C^{ht}(g) = \sum_{k \in K} C^{k,ht}(g)$. For the corresponding homogeneous equilibrium flow f, i.e. the instance with diversity parameter $\bar{r}^k$, where $\bar{r}^k$ denotes the average diversity parameter for Commodity k, players of Commodity k share the same cost $c^{\bar{r}^k}(f)$. Then, the *homogeneous total cost* of Commodity k under f is $C^{k,hm}(f) = d_k c^{\bar{r}^k}(f)$, and the *homogeneous total cost* of f is $C^{hm}(f) = \sum_{k \in K} C^{k,hm}(f)$.

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[1]This is basically because the flow g_p of a path p with diversity parameter within some range, can be changed to have parameter equal to the average parameter on p, and the equilibrium remains.
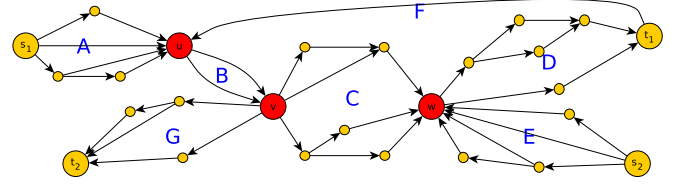

Figure 1: A block-matching network of 2 commodities. G_1 and G_2 are series-parallel and their block representations are $G_1 = s_1 A u B v C w D t_1$ and $G_2 = s_2 E w D t_1 F u B v G t_2$. G_1 and G_2 share exactly blocks B and D and do not share any edge on any other of their blocks. If we add an edge from s_1 to t_1, then the network stops being block-matching since G_1 will be a block by itself and it will not match any of the blocks of G_2.

Finally, if $C^{ht}(g) \leq C^{hm}(f)$, we say that *diversity helps*; if not, we say that *diversity hurts*. For our characterization to be meaningful, we assume an *average-respecting* demand, i.e., a demand where $\forall i, j : \bar{r}^i = \bar{r}^j$. Otherwise, diversity may hurt in simple instances, e.g., with two parallel links and two commodities.

Networks. For a network G we let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively.

A directed s–t network G is *series-parallel* if it consists of a single edge (s, t), or it is formed by the series or parallel composition of two series-parallel networks with terminals (s_1, t_1) and (s_2, t_2), respectively. In a *series composition*, t_1 is identified with s_2, s_1 becomes s, and t_2 becomes t. In a *parallel composition*, s_1 is identified with s_2 and becomes s, and t_1 is identified with t_2 and becomes t. The internal vertices of a series-parallel network G are all its vertices other than its terminals.

An s–t series-parallel network may be represented using a sequence of networks B_j connected in series, where each B_j is either a single edge or two series-parallel networks connected in parallel. Given a series-parallel network H, we can write $H = sB_1 v_1 B_2 v_2 \ldots B_{b-1} v_{b-1} B_b t$, where for any j and triple $x B_j y$, x and y are the terminals of the series-parallel network B_j, and B_j is either a single edge or a parallel combination of two series-parallel networks. We refer to the B_j's as *blocks*, the prescribed representation as the *block representation* of H, and the v_i's as *separators*, as they separate s from t. Two series-parallel networks G_1 and G_2 are said to be *block-matching* if for every block B of G_1 and every block D of G_2, either $E(B) = E(D)$ or $E(B) \cap E(D) = \emptyset$. Note that $E(B) = E(D)$ implies that B and D have the same terminals and direction, as for either B or D, the source vertex will have only outgoing edges toward the internal vertices and the target vertex will have only incoming edges from the internal vertices.

For a k-commodity network G, let G_i be the subnetwork of G that contains all the vertices and edges of G that belong to a simple s_i–t_i path for Commodity i. In other words, G_i is the subnetwork of G for Commodity i that equilibria flows will consider, as they are, WLOG, acyclic. A multi-commodity network G is *block-matching* if for every i, G_i is series-parallel, and for every i, j, G_i and G_j are block-matching. An example is given in Figure 1.

3 Topology of Single-Commodity Networks for which Diversity Helps

In this section, we fully characterize the topology of single-commodity networks for which, with any choice of heterogeneous demand and edge functions, diversity helps. WLOG we may restrict our attention to single-commodity networks whose edges all belong to some simple source-destination path as only these edges are going to be used by the (WLOG, acyclic) equilibria and thus all other edges can be discarded. It turns out that this topology is exactly that of series-parallel networks (Theorems 1 and 2).

3.1 Series Parallel Implies Diversity is Helpful

Throughout this section we will be considering a heterogeneous instance $\mathcal{G}$ on an s–t series-parallel network G. We let $\mathcal{F}$ denote the corresponding homogeneous instance. We let g denote an equilibrium flow for $\mathcal{G}$ and f an equilibrium flow for $\mathcal{F}$. Finally, we let $C^{ht}(g)$ denote the cost of flow g and $C^{hm}(f)$ the cost of flow f. Although redundant, we keep the superscripts as a further reminder of the flow type at hand.

The key observation is that there is a path P used by flow f such that for every edge in P, $f_e \geq g_e$, and hence for any $r \in [0, r_{\max}]$, $c_p^r(f) \geq c_p^r(g)$ (Lemmas 1[2] and 2). We then deduce our result: $C^{ht}(g) \leq C^{hm}(f)$ (Theorem 1).

Lemma 1. *Let G be an s–t series-parallel network and let x and y be flows on G that route d_1 and d_2 units of traffic respectively, with $d_1 \geq d_2$ and $d_1 > 0$. Then, there exists an s–t path P such that for all $e \in P, x_e > 0$ and $x_e \geq y_e$.*

Lemma 2. *There exists a path P used by f such that for any $r \in [0, r_{\max}]$, $c_p^r(g) \leq c_p^r(f)$.*

Proof. Flows f and g have the same magnitude on the series-parallel network G. Applying Lemma 1 with $x = f$ and $y = g$ implies that there exists an s–t path P such that for all $e \in P, f_e > 0$, implying that WLOG P is used by f, and $f_e \geq g_e$. By assumption, for any $r \in [0, r_{\max}]$, $\ell_e + r\sigma_e$ is non-decreasing, and thus for all $e \in P, \ell_e(f_e) + r\sigma_e(f_e) \geq \ell_e(g_e) + r\sigma_e(g_e)$. Consequently, $\sum_{e \in P} (\ell_e(f_e) + r\sigma_e(f_e)) \geq \sum_{e \in P} (\ell_e(g_e) + r\sigma_e(g_e)) \Leftrightarrow c_p^r(g) \leq c_p^r(f)$ as needed. $\square$

Theorem 1. $C^{ht}(g) \leq C^{hm}(f)$.[3]

Proof. Since G is a series-parallel network, on setting $\bar{r} = E[r]$ and then applying Lemma 2, we obtain that there is a path P used by f such that

$$\ell_p(f) + \bar{r}\sigma_p(f) \geq \ell_p(g) + \bar{r}\sigma_p(g) \qquad (2)$$

WLOG we can assume that the total demand $d = 1$. We first bound the total cost of g in terms of the cost of path P under g and then we use (2) to further bound it in terms of the cost of path P under f. The latter equals the cost of f, as the demand is equal to 1.

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[2]Lemma 1 is similar to [Milchtaich, 2006, Lemma 2].

[3]The inequality might be strict. Consider the case of 2 parallel links with $(\ell_1(x), \sigma_1(x)) = (1, x)$ and $(\ell_2(x), \sigma_2(x)) = (2, 0)$, and 1 unit of flow, half with $r = 0$ and half with $r = 2$.
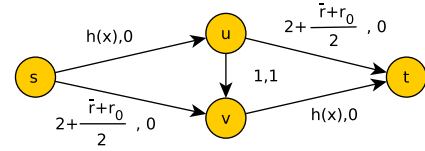

Figure 2: The Braess network of Proposition 1.

Consider the heterogeneous equilibrium flow g. By the equilibrium conditions, for any player of diversity parameter r, for any r, the cost she incurs with flow g is $c^r(g) \leq \sum_{e \in p} \ell_e(g_e) + r \sum_{e \in p} \sigma_e(g_e)$. In other words, there is no incentive to deviate to path P (if not already on it). Thus, for the demand vector $(d_1, \ldots, d_k)$ of diversity parameters $(r_1, \ldots, r_k)$, $C^{ht}(g) \leq \sum_{i=1}^{k} d_i(\sum_{e \in p} \ell_e(g_e) + r_i \sum_{e \in p} \sigma_e(g_e)) = \ell_p(g) + \bar{r}\sigma_p(g)$, with the equality following as the total demand is 1 and the average diversity parameter is $\bar{r} = \sum_{i=1}^{k} d_i r_i$. As P is used by f, we have $C^{hm}(f) = \ell_p(f) + \bar{r}\sigma_p(f)$, and applying (2) we obtain $C^{ht}(g) \leq \ell_p(g) + \bar{r}\sigma_p(g) \leq \ell_p(f) + \bar{r}\sigma_p(f) = C^{hm}(f)$. $\square$

3.2 The Series Parallel Condition is Necessary

To prove the necessity of the network being series-parallel, we begin by constructing an instance for which diversity hurts, i.e. the heterogeneous equilibrium has total cost strictly greater than the total cost of the homogeneous equilibrium (Proposition 1). Then, in Theorem 2, we show how to embed this instance into any network that is not series-parallel.

Recall the Braess graph G_B, shown in Figure 2.

Proposition 1. *For any strictly heterogeneous demand on the Braess graph G_B, there exist edge functions $(\ell_e)_{e \in E}$ and $(\sigma_e)_{e \in E}$ that depend on the demand, for which $C^{ht}(g) > C^{hm}(f)$.* [4]

Proof. We may assume WLOG that the demand is of unit size. Let $\bar{r}$ be the average diversity parameter, r_0 be the minimum of the diversity parameters' distribution and let d_0 be the total demand with diversity parameter equal to r_0. Since the demand is strictly heterogeneous, it must be $r_0 < \bar{r}$. In addition, we let h be any continuous, strictly increasing cost function with $h(\frac{1}{2}) = 1$ and $h(\frac{1}{2} + \frac{d_0}{2}) = 1 + \frac{\bar{r}-r_0}{2}$.

Consider the Braess graph $G_B = (\{s, u, v, t\}, \{(s, u), (u, t), (u, v), (s, v), (v, t)\})$ with cost functions $\ell_{(s,u)}(x) = \ell_{(v,t)}(x) = h(x)$, $\sigma_{(s,u)}(x) = \sigma_{(v,t)}(x) = 0$, $\ell_{(u,t)}(x) = \ell_{(s,v)}(x) = 2 + \frac{\bar{r}+r_0}{2}$, and $\sigma_{(u,t)}(x) = \sigma_{(s,v)}(x) = 0$, and $\ell_{(u,v)}(x) = 1$ and $\sigma_{(u,v)}(x) = 1$. The instance is shown in Figure 2.

The heterogeneous equilibrium g routes the d_0 units that have $r = r_0$ through the zig-zag path, i.e. path s, u, v, t; rest of the flow is split between the upper and lower paths s, u, t and s, v, t, and has $C^{ht}(g) = 3 + \bar{r}$. The homogeneous equilibrium f splits the flow between the upper and lower paths and has $C^{hm}(f) = 3 + \frac{\bar{r}+r_0}{2} < 3 + \bar{r} = C^{ht}(g)$. $\square$

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[4]The proposition remains true even if we are restricted to only using affine functions.

Theorem 2. *If G is not series-parallel, then for any strictly heterogeneous demand there are cost functions for which $C^{ht}(g) > C^{hm}(f)$.*

Proof sketch. If G is not series-parallel then the Braess graph can be embedded in it (see e.g. [Valdes *et al.*, 1979]). Thus, starting from the Braess network G_B, by subdividing edges, adding edges and extending one of the terminals by one edge, we can obtain G. Fixing such a sequence of operations, for the given heterogeneous demand, we start from the Braess instance given by Proposition 1 and apply the sequence of operations one by one, so that, throughout the procedure, there are exactly 3 paths that correspond to the upper, lower and zig-zag paths of Proposition 1, with corresponding costs, and all other paths have some very large costs. Then, $C^{ht}(g) > C^{hm}(f)$ follows exactly as in Proposition 1. □

4 Topology of Multi-Commodity Networks for which Diversity Helps

In this section we fully characterize the topology of multi-commodity networks for which, with any choice of heterogeneous average-respecting demand and edge functions, diversity helps. Because of Theorem 2, if we require diversity to help on any instance on G, then for any commodity i, G_i needs to be series-parallel. Yet, as we shall see in Proposition 2, this is not enough. We also need to understand the overlaps of the G_i's. It turns out that the allowable overlaps are exactly captured by the topology of block-matching networks (Theorems 3 and 4).

4.1 Sufficiency

Using Theorem 1, we can obtain an analogous theorem for the multi-commodity case.

Theorem 3. *Let G be a k-commodity block-matching network. Then, for any instance on G with average-respecting demand $C^{ht}(g) \leq C^{hm}(f)$.*

Proof sketch. Consider Commodity i and let $G_i = s_i B_1 v_1 \dots v_{b_i-1} B_{b_i} t_i$ be its block representation. Consider an arbitrary B_j with terminals v_{j-1} and v_j. Because G is block-matching, any other Commodity l either contains B_j as a block in its block representation or contains none of its edges. Also, recall that, as explained in the preliminaries section, if G_l contains B_j, it has the same terminals v_{j-1} and v_j. This implies that under any routing of the demand, either all of l's demand goes through B_j or none of it does. This means that under both equilibria g and f, the total traffic routed from v_{j-1} to v_j through B_j is the same which further implies that, if restricted to the block, the cost of the heterogeneous equilibrium is less than or equal to that of the homogeneous equilibrium: $C^{ht}(g)\big|_{B_j} \leq C^{hm}(f)\big|_{B_j}$. On the other hand, if we let $\mathcal{B}$ be the set of all the blocks of all commodities, then $C^{ht}(g) = \sum_{B \in \mathcal{B}} C^{ht}(g)\big|_B$ and $C^{hm}(f) = \sum_{B \in \mathcal{B}} C^{hm}(f)\big|_B$ which using the previous inequality proves the result. □
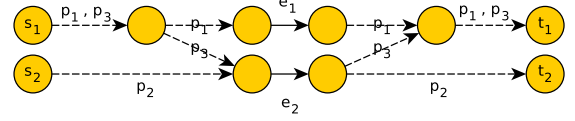

Figure 3: The network for Proposition 2

4.2 Necessity

To derive the necessity we first give an example of a non-block-matching network for which diversity hurts (Proposition 2). Then, after giving some properties for commodities for which the corresponding G_i's are series-parallel (Lemmas 3 and 4), we mimic the above example to obtain contradicting instances for networks that are not block-matching and thereby prove Theorem 4.

Let G be the following 2-commodity network, depicted in Figure 3. G_2, the subnetwork for Commodity 2, consists of a simple s_2–t_2 path P_2, while G_1, the subnetwork for Commodity 1, is formed from two simple s_1–t_1 paths named P_1 and P_3; P_1 and P_2 are disjoint, while P_2 and P_3 share a single edge, named e_2. Finally e_1 is an edge on P_1 but not on P_3.

Proposition 2. *There exist edge functions and demands on G for which diversity hurts.*

Proof. Let $d_1 = d_2 = 1$ be the total demands for Commodities 1 and 2 respectively. Let G_1's demand consist of $\frac{3}{4}$ single-minded players (i.e., with diversity parameter equal to 0) and $\frac{1}{4}$ players with diversity parameter equal to 4, and let G_2's demand be homogeneous with diversity parameter equal to 1. To all edges other than e_1 and e_2, assign latency and deviation functions equal to 0. Assign edge e_1 the constant latency function $\ell_1(x) = 1$ and the constant deviation function $\sigma_1(x) = 2$. Assign edge e_2 the constant deviation $\sigma_2 = 0$, and as latency function any ℓ_2 that is continuous and strictly increasing, with $\ell_2(1) = 3$ and $\ell_2(\frac{5}{4}) = 9$.

The equilibrium costs depend only on the flow through edges e_1 and e_2, as all other edges have cost 0. Also note that at least 1 unit of flow will go through e_2 as this is the only route for G_2's demand.

In the heterogeneous equilibrium g of this instance, $\frac{3}{4}$ units of flow are routed through e_1, and $1 + \frac{1}{4}$ units of flow are routed through e_2, giving $C^{ht}(g) = 1 \cdot \frac{3}{4} d_1 + 9 \cdot \frac{1}{4} d_1 + 9 \cdot d_2 = 12$. In the homogeneous equilibrium f, G_1's demand uses only P_1. Thus, $C^{hm}(f) = 3 \cdot d_1 + 3 \cdot d_2 = 6 < C^{ht}(g)$. □

Remark 1. *Proposition 2 would still hold if the common portion of P_1 and P_3, and the portion of P_2 after e_2, both had positive costs instead of zero. This is close to the way we will mimic this instance in the proof of Theorem 4. The idea, in both equilibria, is to route all the flow of Commodity 1 through two paths, P_1 and P_3, each containing one of e_1 or e_2, and to route the flow of Commodity 2 through a path, P_2, that contains e_2. This is done by putting (relatively) big constants as latency functions on all other edges that depart from vertices of the corresponding paths up to the point where e_1 or e_2 is reached, though some care is needed. Then, the relation of the equilibria costs will follow as in Proposition 2, as the exact same edge functions will be used for edges e_1*

and e_2. This will be specified precisely when we give the construction.

Next, we state some useful properties of series-parallel networks that are based on their block structure. They will be used in the proof of Theorem 4.

Lemma 3. *Let i be a commodity of network G and suppose that G_i is series-parallel.*

(i) Let B_1 and B_2 be distinct blocks of G_i, with B_1 preceding B_2. There is no edge in G from an internal vertex of B_1 to an internal vertex of B_2.

(ii) Let u and v be vertices in G_i. If (u, v) is an edge of G then there is a simple s_i–t_i path in G_i that contains both u and v (not necessarily in that order).

Lemma 4. *Let i be a commodity of network G and suppose that G_i is series-parallel with block representation $G_i = s_i B_1 v_1 \ldots v_{b-1} B_b t_i$. Let w be a vertex of B_k for some $k \in [b]$.*

(i) Suppose that $w \neq v_{k-1}$, and let P be an arbitrary path from a vertex u, in a block A that precedes B_k in the block representation, to vertex w. Let w' be the first vertex on P that is an internal vertex in B_k, if any. Then P must include an edge of B_k exiting v_{k-1} prior to visiting w'.

(ii) Suppose that $w \neq v_k$. Then any path of G from w to a vertex u in a block succeeding B_k has to first enter v_k through one of its incoming edges that belong to B_k, before going to a block C that succeeds B_k in the block representation.

(iii) Every simple v_{k-1}–v_k path in G is completely contained in B_k.

Theorem 4. *Let G be a multi-commodity network. If diversity helps for every instance on G with average-respecting demand (i.e. for any heterogeneous equilibrium g and any homogeneous equilibrium f, $C^{ht}(g) \leq C^{hm}(f)$), then G is a block-matching network.*

Proof. Let G have k commodities. First, we note that for any $i \in [k]$, G_i is a series-parallel network. For otherwise, by Proposition 1, there is some heterogenous players' demand for Commodity i and edge functions for G_i such that diversity hurts. Letting all other commodities have zero demand yields an instance on G for which diversity hurts, a contradiction.

To prove that G is block-matching, it remains to show that for any two commodities i and j of G, for any block B of G_i and any block D of G_j, either $E(B) = E(D)$ or $E(B) \cap E(D) = \emptyset$. To reach a contradiction we assume otherwise, i.e. WLOG we assume that for Commodities 1 and 2 there exist two blocks B of G_1 and D of G_2 that share some common edge, and at the same time, WLOG, there is an edge in B that is not in D. The latter implies that B is not a single edge, and thus it must be a parallel combination of two series-parallel networks.

Let u and v be the endpoints of B. We first prove that all simple s_2–t_2 paths of G_2 that share an edge with B first traverse an edge starting at u before traversing any other edge of B (Proposition 3). Then we prove that all s_2–t_2 simple paths of G_2, that share an edge with B, reach u before traversing any internal vertex of B (Proposition 4). Since $E(B) \cap E(D) \neq \emptyset$, there is a simple s_2–t_2 path of G_2 that shares an edge with B. Proposition 4 implies that this path, Q,
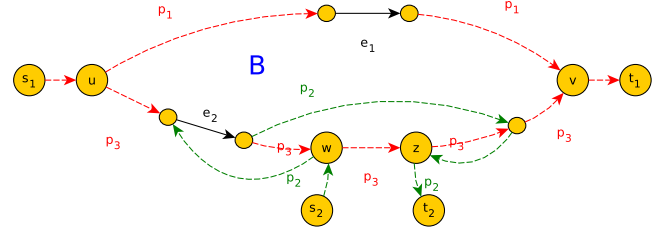

Figure 4: Illustrating why $P_2 \neq P$ in general in Proposition 3

has a subpath consisting of a simple s_2–u path Q_1 that shares no internal vertex with B. A completely symmetric argument shows that Q has a subpath consisting of a simple v–t_2 path Q_3 that shares no internal vertex with B. [5] But then, for any simple u–v path Q_2 inside B, the path $Q' = Q_1, Q_2, Q_3$ is a simple s_2–t_2 path, and thus it belongs to G_2. But this implies that all the edges of B belong to G_2 and because B is a block, these edges will all be in a single block of G_2. This block must be block D, since by assumption $E(B) \cap E(D) \neq \emptyset$, contradicting the existence of an edge in B and not in D. Therefore, once these propositions are proved, the theorem will follow.

The proofs of these propositions rely on the same idea. For each proposition, assuming that it does not hold, we construct instances, i.e. we choose demand and edge functions for G, such that diversity hurts, contradicting the assumption that for any instance on G diversity helps. The construction of the contradicting instances is based on Remark 1.

Proposition 3. *Let P be a simple s_2–t_2 path in G_2 which shares an edge with B. The first edge on P in B departs from u, i.e. has the form (u, x) for some x in B.*

Proof. Let B be the parallel combination of H_1 and H_2. WLOG we may assume that P only visits vertices of G_1, plus s_2 and t_2, as we may treat subpaths of P that have vertices that lie outside G_1 as simple edges. Let w be the first internal vertex of P that belongs to B, and WLOG suppose that w lies in H_1. By Lemma 3(ii), the edge of P exiting w will either go toward t_1, i.e. forward, and thus traverse an edge of B for the first time (recall also Lemma 4(ii)), or will go toward s_1, i.e. backward, either staying in H_1 or going back to one of the preceding blocks of B. If it goes to one of the preceding blocks of B, then by Lemma 4(i), it has to traverse an edge of B departing from u in order to re-enter the internal portion of B (recall that P has some edge in B) and then the proposition would hold. The remaining possibility is that the backward edge leads to another internal vertex of H_1. However, we can only repeat this process finitely often so if the proposition does not hold, it must be that P eventually traverses a first edge in B that departs from an internal vertex of H_1. In this case we will reach a contradiction by creating an instance where diversity hurts. This instance will be based on the instance of Proposition 2.

We would like to use the following construction at this point. Let P_2 be the path P resulting from the discussion in

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[5]For the symmetric argument, simply reverse all the arcs and the directions of the demand.

the previous paragraph and let e_2 be the first edge on P that lies in B. Then let P_3 be an s_1–t_1 path through e_2. Recall that e_2 lies in H_1. Now let P_1 be an s_1–t_1 path that goes through H_2 and let e_1 be an arbitrary edge on P_1 in B. The intention is to force the s_1–t_1 flow to use just paths P_1 and P_3, while the s_2–t_2 flow uses just path P_2, at the same time ensuring that diversity is harmful as in Proposition 2. Consider the following edge functions. e_1 and e_2 receive the same edge functions as in Proposition 2. The other edges all receive a constant deviation function equal to 0. For their latency functions, edges on P_1 and P_3 that are in B receive 0 functions. Out-edges from P_1 and P_3 that lie in B all receive functions of constant value N even if they are on P_2. All as yet unassigned edges on P_2 receive 0 functions, and the remaining edges are all given functions of constant value $M \gg N$. However, the example in Figure 4 shows that there is a zero cost s_2–t_2 path (s_2, w, z, t_2), which defeats the construction.

We fix this problem by defining the path P_2 as follows. Let x be the first vertex on path P (in the example, this is w) such that there is an edge (x, y) in B and such that there is a y–t_2 path P_{y-t_2} which does not go through any earlier vertex on P (i.e. any vertex from s_2 to x inclusive). Then (x, y) is chosen to be e_2, and P_2 is defined to be the simple path comprising the initial portion of P up to x, followed by e_2, followed by P_{y-t_2} (it may be that $P_2 = P$). Now the above cost functions, modulo a few details, achieve the desired contradiction. $\square$

Proposition 4. *All simple s_2–t_2 paths of G_2 that share an edge with B reach u before any internal vertex of B.*

The proof is largely similar to that of Proposition 3. $\square$

Acknowledgements

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When Does Diversity of User Preferences Improve Outcomes in Selfish Routing?[⋆]

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Abstract. We seek to understand when heterogeneity in user preferences yields improved outcomes in terms of overall cost. That this might be hoped for is based on the common belief that diversity is advantageous in many settings. We investigate this in the context of routing. Our main result is a sharp characterization of the network settings in which diversity always helps, versus those in which it is sometimes harmful.

Specifically, we consider routing games, where diversity arises in the way that users trade-off two criteria (such as time and money, or, in the case of stochastic delays, expectation and variance of delay). Our main contributions—a conceptual and a technical one— are the following:

1) A participant-oriented measure of cost in the presence of user diversity, together with the identification of the natural benchmark: the same cost measure for an appropriately defined average of the diversity.

2) A full characterization of those network topologies for which diversity always helps, for all latency functions and demands. For single-commodity routings, these are series-parallel graphs, while for multi-commodity routings, they are the newly-defined "block-matching" networks. The latter comprise a suitable interweaving of multiple series-parallel graphs each connecting a distinct source-sink pair.

While the result for the single-commodity case may seem intuitive in light of the well-known Braess paradox, the two problems are different: there are instances where diversity helps although the Braess paradox occurs, and vice-versa. But the main technical challenge is to establish the "only if" direction of the result for multi-commodity networks. This follows by constructing an instance where diversity hurts, and showing how to embed it in any network which is not block-matching, by carefully exploiting the way the simple source-sink paths of the commodities intersect in the "non-block-matching" portion of the network.

1 Introduction

People are inherently diverse and it is a common belief that diversity helps. In one of the central themes of algorithmic game theory—the tension between selfish behavior and social optimality—can diversity of user preferences help to bring us closer to the coveted social optimality? We provide an answer to this question in the context of non-atomic selfish routing, where diversity naturally arises in the way users trade-off two criteria, for example, time and money, or, in the presence of uncertain delays, expectation and variance of delay.

Diversity is reflected in combining the two criteria via different individual coefficients, which we call the 'diversity parameters'. We consider a linear combination of the two criteria, as in the literature on tolls where users minimize travel time plus tolls, e.g., [Beckmann *et al.*, 1956; Fleischer *et al.*, 2004] or the literature on risk-averse selfish routing where users minimize expected travel time plus variance ([Nikolova and Stier-Moses, 2014]), or more generally travel time plus a deviation function ([Kleer and Schäfer, 2016]).

We are interested in understanding whether heterogeneity in user preferences improves collective outcomes or makes them worse. As we shall see, there is no unique answer. Rather, it depends on the setting. To address our question we need to specify how to measure the cost of an outcome, and define our comparison point, namely a benchmark setting with no user heterogeneity. As explained above, to measure the cost of an outcome, we treat a user's cost as the sum of two terms associated with two criteria: If we let ℓ_P denote the cost of one criterion (e.g., the latency) over a path P, and v_P be the cost of the second criterion, then

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the overall cost is given by $\ell_P + r \cdot v_P$, where r is our diversity parameter. The special case of $r = 0$ corresponds to indifference to the second criterion and results in the classic selfish routing model where users simply minimize travel time.

A first approach to measuring the effect of diversity might be to compare the cost of an outcome with $r = 0$ (i.e., just the total latency) to that with other values of r, including possibly mixed values of r across the population being routed. However, this approach does not pinpoint the gains and losses from user heterogeneity as opposed to user homogeneity; rather, it (mostly) pinpoints the gains and losses depending on whether players are affected by the second criterion or not. Instead, we focus on the sum of the costs incurred by the users as measured by their cost functions, and compare costs incurred by a heterogeneous population of users to those incurred by an equivalent population of homogeneous users. What are equivalent populations? Suppose the heterogeneous population's diversity profile is given by a population density function $f(r)$. Then, we define the corresponding homogeneous population to have the single diversity parameter $\bar{r} = \int r f(r) dr$. In addition, we require the two populations to have the same size, in the sense that the total source-to-sink flows that they induce are equal.

For this setting, we completely characterize the graphs for which user heterogeneity does no harm, in the sense of only reducing the total cost as perceived by users, for both the case of a single commodity (i.e. one source-to-sink flow) and for multiple commodities (i.e. flows between multiple source-sink pairs). For the single-commodity case, these graphs are exactly the series-parallel graphs, with the source being the "start" vertex of the graph, and the sink being the "terminal" vertex. For the multi-commodity case, each commodity flows over a series-parallel graph, and further these subgraphs need to overlap in a particular manner which we specify later, in the formal statement of results. For all other graphs, we provide examples of desired source-to-sink demands for which the resulting equilibrium flows are more expensive than the flows in the corresponding homogeneous problem.

Related work. To the best of our knowledge this is the first work that methodically compares the effects of heterogeneity and homogeneity in user preferences for a self-interested routing population. In fact, in the broader area of algorithmic game theory, this seems to be the first time that a question of this type has been considered, with the exception of Chen *et al.* [2014]. [1] Algorithmic game theory research mentioning diversity exists in the context of the theory of evolution (e.g., Mehta *et al.* [2015]; Chastain *et al.* [2013]), which is very different from our focus.

Since we provide attitudes to time versus money and time versus risk as motivating examples for user diversity, we briefly mention related work on tolls and on risk-averse selfish routing. Regarding the latter, there are multiple ways to model how the behavior of players incorporates risk when uncertainty is present (see e.g. Rockafellar [2007]). Piliouras *et al.* [2013] studied the effect that different risk attitudes may have on a system's performance at equilibrium. They did so by examining the price of anarchy (i.e. the ratio of the cost at equilibrium to the optimal cost) for different risk formulations. Nikolova and Stier-Moses [2015] and Lianeas *et al.* [2016] studied the degradation of a network's performance due to risk aversion. This kind of degradation is captured by the price of risk aversion, which compares the cost of the equilibrium when players are risk-averse to the equilibrium cost when players are risk-neutral. The above works assumed that all players have the same risk averse preferences, which we call risk homogeneity; they do not offer any indication as to whether and under what circumstances risk heterogeneity improves or harms a system's performance. In contrast, Fotakis *et al.* [2015] considered games with heterogeneous risk-averse players. They showed how uncertainty may and can be used to improve a network's performance, but the effects of heterogeneity as opposed to homogeneity were left unexamined. Regarding the related literature on tolls, early results (e.g., Beckmann *et al.* [1956]) showed that tolls can help implement the social optimum as an equilibrium, when users all have the same linear objective function combining time and money. Much more

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[1] In a different setting, Chen *et al.* [2014] show how diversity may affect a bound they prove for the price of anarchy on parallel link networks.

recently, these results were extended to the case where users trade-off travel time and money differently, by Cole *et al.* [2003] and Fleischer [2005] for the single commodity case, and by Karakostas and Kolliopoulos [2004b] and Fleischer *et al.* [2004] for the multicommodity case. We remark that in the above works, apart from Cole *et al.* [2003], the social welfare is defined as the total travel time, whereas in our work we consider the total user cost, which encapsulates both criteria. This, for example, is also the case for Christodoulou *et al.* [2014] and Karakostas and Kolliopoulos [2004a]. We further note that if the social welfare is defined as the total travel time then there are simple instances on parallel link graphs where diversity is harmful.

Characterizing the topology of networks that satisfy some property is a common theme in computer science. Relevant to our work, Epstein *et al.* [2009] characterized the topology of single-commodity networks for which all Nash equilibria are social optima (under bottleneck costs), and Milchtaich [2006] characterized the topology of single-commodity networks which do not suffer from the Braess Paradox for any cost functions. Chen *et al.* [2015] fully characterized the topology of multi-commodity networks that do not suffer from the Braess paradox. These characterizations appear similar to ours, although there does not seem to be any other connection between the two problems, as (i) there are instances where diversity helps while the Braess Paradox occurs and others where diversity hurts but the paradox does not occur, and (ii) the Braess Paradox may occur in series-parallel networks when considering selfish routing with heterogeneity in user preferences, which is not the case for the classic selfish routing model.

In the following works, the characterizing topology for the corresponding question (for a single commodity) is similar to ours. Fotakis and Spirakis [2008] considered atomic games and proved that series-parallel networks are the largest class of networks for which strongly optimal tolls are known to exist. Nikolova and Stier-Moses [2015] considered homogeneous agents and a social cost function that does not account for the second criterion; They showed that series-parallel networks admit the best bound on the degradation of the network due to risk aversion. Theorem 4 of Acemoglu *et al.* [2016], proves that series-parallel networks are the characterizing topology for what they call the Informational Braess Paradox with Restricted Information Sets; this theorem compares the cost of one agent type before and after more information is revealed to agents of that type, but does not consider the change in the cost of other agent types. In contrast, our work considers non-atomic games with heterogeneous agents and bounds the overall costs faced by the collection of agents. Most relevant to our work is Meir and Parkes [2014] and its Theorem 3.1 as it implies that for series-parallel networks the cost of an agent of average parameter only increases when switching from the heterogeneous instance to the corresponding homogeneous one and thus for our sufficiency theorems, one is left to prove that the heterogeneous equilibrium cost is no greater than the cost of an agent of average parameter (though we give a different proof).

Contribution. We fully characterize the topology of networks for which diversity is never harmful, regardless of the demand size and the distribution of the diversity parameter (discrete or continuous). We do so both for single and multi-commodity networks.

For single-commodity networks it turns out that this topology is that of series-parallel networks. In Theorem 1, we show that if the network is series-parallel, then diversity only helps for any choice of demand and edge functions. The key observation is that there is a path for which the homogeneous flow is at least as large as the heterogeneous flow. As the cost of the homogeneous flow is the same on all used paths, while the cost of each unit of heterogeneous flow is lowest on the path it uses, one can then deduce that the cost of the heterogeneous flow is at most that of the homogeneous flow. To show necessity, we first provide an instance on the Braess graph for which diversity is harmful, and then show how to embed it in any non-series-parallel graph.

In multi-commodity networks, by the result above, each commodity must route its flow through a series-parallel subnetwork. But, as Proposition 2 shows, this is not enough, and the way in which these series-parallel networks overlap needs to be constrained. The necessary constraint is exactly captured by the class

of *block-matching* networks, defined in this paper. Sufficiency in this case then follows quite easily from the same result for the single commodity case.

The main technical challenge is to show necessity. To this end, assuming diversity does no harm, we show, via a case analysis, how the subnetworks of the commodities may overlap. First, in Proposition 2, we give an instance on a network of two commodities and three paths for which diversity hurts. Then we mimic this instance on a general network. The difficult part is to choose the corresponding paths for the mimicking, so that, in the created instance, all the flow under both equilibria goes through these paths. The challenge is that the commodities' subnetworks may overlap in subtle ways.

2 Preliminaries

We consider a directed multi-commodity network $G = (V, E)$ with an aggregate demand of d_k units of flow between origin-destination pairs (s_k, t_k) for $k \in K$. We let $\mathcal{P}_k$ be the set of all paths between s_k and t_k, and $\mathcal{P} := \cup_{k \in K} \mathcal{P}_k$ be the set of all origin-destination paths. We let $[m]$ denote $\{1, \ldots, m\}$. We assume that $K = [m]$, for some m. The users in the network—i.e., the players of the game—must choose routes that connect their origins to their destinations. We encode the collective decisions of users in a flow vector $f = (f_\pi)_{\pi \in \mathcal{P}} \in \mathbb{R}_+^{|\mathcal{P}|}$ over all paths. Such a flow is feasible when demands are satisfied, as given by constraints $\sum_{\pi \in \mathcal{P}_k} f_\pi = d_k$ for all $k \in K$. For simplicity, we let f_e denote the flow on edge e; note that $f_e = \sum_{\pi:e \in \pi} f_\pi$. When we need multiple flow variables, we use the analogous notation g, g_π, g_e.

The network is subject to congestion that affects two criteria the players consider. These two criteria are modeled by two *edge-dependent functions* that take as input the edge flow f_e of e, for each edge e: a latency function $\ell_e(x)$ assumed to be continuous and non-decreasing, and a deviation function $\sigma_e(x)$ assumed to be continuous (but not necessarily non-decreasing). Function $\ell_e(\cdot)$ represents the first criterion while $\sigma_e(\cdot)$ represents the second criterion.

Players choose paths according to a linear combination of the first criterion and the second criterion along the route. Throughout the paper we refer to the players' objective as the cost along a route. Formally, for a given user, on letting $\ell_\pi(f) = \sum_{e \in \pi} \ell_e(f_e)$ and $\sigma_\pi(f) = \sum_{e \in \pi} \sigma_e(f_e)$, for a constant $r \geq 0$ that quantifies the user diversity parameter, the user's cost along route π under flow f is

$$c_\pi^r(f) = \sum_{e \in \pi} \ell_e(f_e) + r \sum_{e \in \pi} \sigma_e(f_e) = \ell_\pi(f) + r\sigma_\pi(f) \tag{1}$$

We assume that for any edge and for any player's diversity parameter r, the functions ℓ_e and $\ell_e + r\sigma_e$ are non-decreasing. We note that if there is an upper bound $r_{\max}$ on the possible values of the diversity parameter r, then the latter assumptions do not require σ_e to be non-decreasing. This is desirable because, for example, in risk-averse selfish routing where σ_e models the variance, σ_e can be a decreasing function of the flow.

Players Heterogeneity. We assume that the diversity parameter distribution r_k for each commodity k is heterogeneous, i.e. there may be more than one value of the diversity parameters r for the players routing commodity k. We use the term *single-minded* to refer to players with $r = 0$.

We consider two cases: where the distribution of the diversity parameter among the players is discrete and where it is continuous.[2] For a discrete distribution of, say, n discrete values $r_1^k, \ldots, r_n^k$, the demand d_k is a vector $d_k = (d_1^k, \ldots d_n^k)$ where each d_i^k denotes the total demand of Commodity k with diversity parameter r_i^k. We let d^k denote Commodity k's total demand, $d^k = \sum_{i=1}^n d_i^k$. For a continuous distribution with infimum and supremum $r_{\min}^k$ and $r_{\max}^k$ respectively, the demand d_k is given by a density function $\rho_k : [r_{\min}, r_{\max}] \to \mathbb{R}_{\geq 0}$ such that for any two values $r_1 \leq r_2$ the total demand with diversity parameter

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[2] In fact, for our results, we could only focus on the discrete case, though we would first have to prove that any continuous case instance has a corresponding discrete case instance such that their homogeneous equilibria have the same costs, as do their heterogeneous equilibria.

$r_1 \leq r \leq r_2$ has magnitude $\int_{r_1}^{r_2} \rho_k(r)dr$, and $d^k = \int_{r_{\min}^k}^{r_{\max}^k} \rho_k(r)dr$. Variables f_π^r and f_e^r denote the flow of diversity parameter r on path π and edge e, respectively.

Formally, an instance is described by the tuple $(G, \{(\ell_e, \sigma_e)\}_{e \in E}, \{(s_k, t_k)\}_{k \in K}, \{d_k\}_{k \in K}, \{r_k\}_{k \in K})$ for the discrete case, where $r_k = (r_1^k, \ldots, r_n^k)$ is the vector of different diversity parameters encountered in the heterogeneous population, and by the tuple $(G, \{(\ell_e, \sigma_e)\}_{e \in E}, \{(s_k, t_k)\}_{k \in K}, \{\rho_k\}_{k \in K})$ in the continuous case.

Equilibrium flows. The Wardrop equilibrium of an instance is a flow f such that for every $k \in K$, for every path $\pi \in \mathcal{P}_k$ with positive flow, and any diversity parameter r on it, the path cost $c_\pi^r(f) \leq c_{\pi'}^r(f)$ for all paths $\pi' \in \mathcal{P}_k$.

From here on, we shall refer to the Wardrop equilibrium as the equilibrium. Our goal is to compare the total user cost at the equilibrium of an instance with a population that has heterogeneous diversity parameters, to the total user cost at the equilibrium of the same instance but with the population of each commodity keeping its magnitude changed to be homogeneous, with diversity parameter equal to the expected value of the diversity parameter distribution in the heterogeneous population of the commodity. To differentiate more easily, for a heterogeneous instance we call the former the *heterogeneous equilibrium* and the latter the (corresponding) *homogeneous equilibrium*. We usually denote the heterogeneous equilibrium by g and the homogeneous equilibrium by f. The existence of both equilibria is guaranteed by e.g. Schmeidler [1973, Theorem 2]. We note here that in general we do not need uniqueness of equilibria, neither for the edge costs nor for the edge flows. Our results hold for any arbitrary pair of heterogeneous and homogeneous equilibria of the corresponding instances. Also, as for classic routing games, without loss of generality (WLOG) we may assume that equilibrium flows are acyclic.

Total Costs. For a heterogeneous equilibrium flow vector g, in the discrete case, the *heterogeneous total cost* of Commodity k is denoted by $C^{k,ht}(g) = \sum_{j=1\ldots n} d_j^k c^{k,r_j^k}(g)$ where $c^{k,r_j^k}(g)$ denotes the common cost at equilibrium g for players of diversity parameter r_j^k in Commodity k. In the continuous case, the *heterogeneous total cost* of Commodity k is denoted by $C^{k,ht}(g) = \int_{r_{\min}^k}^{r_{\max}^k} \rho_k(r)c^{k,r}(g)dr$, where $c^{k,r}(g)$ denotes the common cost at equilibrium g for players of diversity parameter r in Commodity k. The *heterogeneous total cost* of g is then $C^{ht}(g) = \sum_{k \in K} C^{k,ht}(g)$. For the corresponding homogeneous equilibrium flow f, i.e. the instance with diversity parameter $\bar{r}^k$, where $\bar{r}^k$ denotes the average diversity parameter for Commodity k, players of Commodity k share the same cost $c^{\bar{r}^k}(f)$. Then, the *homogeneous total cost* of Commodity k under f is $C^{k,hm}(f) = d_k c^{\bar{r}^k}(f)$, and the *homogeneous total cost* of f is $C^{hm}(f) = \sum_{k \in K} C^{k,hm}(f)$. Finally, if $C^{ht}(g) \leq C^{hm}(f)$, we say that *diversity helps*; if not, we say that *diversity hurts*. For our characterization to be meaningful, we assume an *average-respecting* demand, i.e., a demand where $\forall i, j : \bar{r}^i = \bar{r}_j$. Otherwise, diversity may hurt in simple instances, e.g., with two parallel links and two commodities (see Appendix A.1 for an example).

Networks. For a network G we let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively.

A directed s–t network G is *series-parallel* if it consists of a single edge (s, t), or it is formed by the series or parallel composition of two series-parallel networks with terminals (s_1, t_1) and (s_2, t_2), respectively. In a *series composition*, t_1 is identified with s_2, s_1 becomes s, and t_2 becomes t. In a *parallel composition*, s_1 is identified with s_2 and becomes s, and t_1 is identified with t_2 and becomes t. The internal vertices of a series-parallel network G are all its vertices other than its terminals.

An s–t series-parallel network may be represented using a sequence of networks B_j connected in series, where each B_j is either a single edge or two series-parallel networks connected in parallel[3]. Given a series-parallel network H, we can write $H = sB_1v_1B_2v_2\ldots B_{b-1}v_{b-1}B_bt$, where for any j and triple xB_jy, x and y are the terminals of the series-parallel network B_j, and B_j is either a single edge or a parallel

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[3] Note that this definition captures the simple case of many edges connected in parallel.

combination of two series-parallel networks. We refer to the B_j's as *blocks*, the prescribed representation as the *block representation* of H, and the v_i's as *separators*, as they separate s from t. Two series-parallel networks G_1 and G_2 are said to be *block-matching* if for every block B of G_1 and every block D of G_2, either $E(B) = E(D)$ or $E(B) \cap E(D) = \emptyset$. Note that $E(B) = E(D)$ implies that B and D have the same terminals and direction, as for either B or D, the source vertex will have only outgoing edges toward the internal vertices and the target vertex will have only incoming edges from the internal vertices.

For a k-commodity network G, let G_i be the subnetwork of G that contains all the vertices and edges of G that belong to a simple s_i–t_i path for Commodity i. In other words, G_i is the subnetwork of G for Commodity i that equilibria flows will consider, as they are, WLOG, acyclic. A multi-commodity network G is *block-matching* if for every i, G_i is series-parallel, and for every i, j, G_i and G_j are block-matching. An example is given in Figure 1.
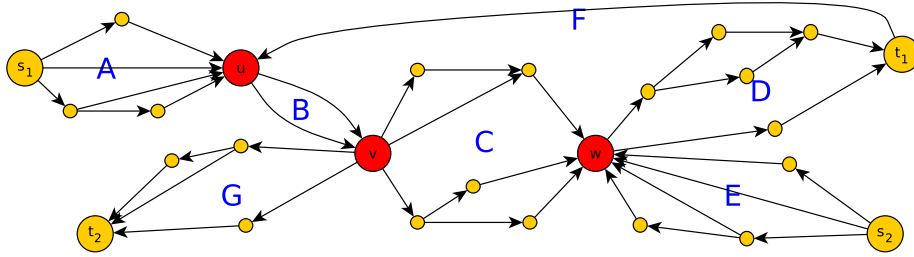

Fig. 1. A block-matching network of 2 commodities. G_1 and G_2 are series-parallel and their block representations are $G_1 = s_1 A u B v C w D t_1$ and $G_2 = s_2 E w D t_1 F u B v G t_2$. G_1 and G_2 share exactly blocks B and D and do not share any edge on any other of their blocks. If we add an edge from s_1 to t_1, then the network stops being block-matching since G_1 will be a block by itself and it will not match any of the blocks of G_2.

3 Topology of Single-Commodity Networks for which Diversity Helps

In this section, we fully characterize the topology of single-commodity networks for which, with any choice of heterogeneous demand and edge functions, diversity helps. WLOG we may restrict our attention to single-commodity networks whose edges all belong to some simple source-destination path as only these edges are going to be used by the (WLOG, acyclic) equilibria and thus all other edges can be discarded. It turns out that this topology is exactly that of series-parallel networks (Theorems 1 and 2). Ommited proofs can be found in Appendix A.

3.1 Series Parallel Implies Diversity is Helpful

Throughout this section we will be considering a heterogeneous instance $\mathcal{G}$ on an s–t series-parallel network G. We let $\mathcal{F}$ denote the corresponding homogeneous instance. We let g denote an equilibrium flow for $\mathcal{G}$ and f an equilibrium flow for $\mathcal{F}$. Finally, we let $C^{ht}(g)$ denote the cost of flow g and $C^{hm}(f)$ the cost of flow f. Although redundant, we keep the superscripts as a further reminder of the flow type at hand.

The key observation is that there is a path P used by flow f such that for every edge in P, $f_e \geq g_e$, and hence for any $r \in [0, r_{\max}]$, $c_p^r(f) \geq c_p^r(g)$ (Lemmas 1[4] and 2). We then deduce our result: $C^{ht}(g) \leq C^{hm}(f)$ (Theorem 1).

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[4] Lemma 1 is similar to [Milchtaich, 2006, Lemma 2], though for completeness we include its proof here.

Lemma 1. *Let G be an s–t series-parallel network and let x and y be flows on G that route d_1 and d_2 units of traffic respectively, with $d_1 \geq d_2$ and $d_1 > 0$. Then, there exists an s–t path P such that for all $e \in P, x_e > 0$ and $x_e \geq y_e$.*

Lemma 2. *There exists a path P used by f such that for any $r \in [0, r_{\max}], c_p^r(g) \leq c_p^r(f)$.*

Proof. Flows f and g have the same magnitude on the series-parallel network G. Applying Lemma 1 with $x = f$ and $y = g$ implies that there exists an s–t path P such that for all $e \in P, f_e > 0$, implying that WLOG P is used by f, and $f_e \geq g_e$. By assumption, for any $r \in [0, r_{\max}], \ell_e + r\sigma_e$ is non-decreasing, and thus for all $e \in P, \ell_e(f_e) + r\sigma_e(f_e) \geq \ell_e(g_e) + r\sigma_e(g_e)$. Consequently, $\sum_{e \in P} \left(\ell_e(f_e) + r\sigma_e(f_e) \right) \geq \sum_{e \in P} \left(\ell_e(g_e) + r\sigma_e(g_e) \right) \Leftrightarrow c_p^r(g) \leq c_p^r(f)$ as needed.

Theorem 1. $C^{ht}(g) \leq C^{hm}(f)$.[5]

Proof. Since G is a series-parallel network, on setting $\bar{r} = E[r]$ and then applying Lemma 2, we obtain that there is a path P used by f such that

$$\ell_p(f) + \bar{r}\sigma_p(f) \geq \ell_p(g) + \bar{r}\sigma_p(g) \tag{2}$$

WLOG we can assume that the total demand $d = 1$. We first bound the total cost of g in terms of the cost of path P under g and then we use (2) to further bound it in terms of the cost of path P under f. The latter equals the cost of f, as the demand is equal to 1.

Consider the heterogeneous equilibrium flow g. By the equilibrium conditions, for any player of diversity parameter r, for any r, the cost she incurs with flow g is $c^r(g) \leq \sum_{e \in p} \ell_e(g_e) + r \sum_{e \in p} \sigma_e(g_e)$. In other words, there is no incentive to deviate to path P (if not already on it). Thus, if the diversity parameters are discrete, given by a demand vector $(d_1, \ldots, d_k)$ of diversity parameters $(r_1, \ldots, r_k)$,

$$C^{ht}(g) \leq \sum_{i=1\ldots k} d_i \left(\sum_{e \in p} \ell_e(g_e) + r_i \sum_{e \in p} \sigma_e(g_e) \right) = \ell_p(g) + \bar{r}\sigma_p(g),$$

with the last equality following as the total demand is 1 and the average diversity parameter is $\bar{r} = \sum_{i=1}^k d_i r_i$. If instead the diversity parameters are continuously distributed on the demand with density function $\rho(r)$, with $r_{\min}$ and $r_{\max}$ being their infimum and supremum respectively,

$$C^{ht}(g) \leq \int_{r_{\min}}^{r_{\max}} \rho(r) \left(\sum_{e \in p} \ell_e(g_e) + r \sum_{e \in p} \sigma_e(g_e) \right) dr = \ell_p(g) + \bar{r}\sigma_p(g),$$

with the last equality following as the total demand is 1, i.e. $\int_{r_{\min}}^{r_{\max}} \rho(r)dr = 1$, and the average diversity parameter is $\bar{r} = \int_{r_{\min}}^{r_{\max}} r\rho(r)dr$. In both the discrete and continuous case, as P is used by f, we have $C^{hm}(f) = \ell_p(f) + \bar{r}\sigma_p(f)$, and applying (2) we obtain

$$C^{ht}(g) \leq \ell_p(g) + \bar{r}\sigma_p(g) \leq \ell_p(f) + \bar{r}\sigma_p(f) = C^{hm}(f).$$

3.2 The Series Parallel Condition is Necessary

To prove the necessity of the network being series-parallel, we begin by constructing an instance for which diversity hurts, i.e. the heterogeneous equilibrium has total cost strictly greater than the total cost of the homogeneous equilibrium (Proposition 1). Then, in Theorem 2, we show how to embed this instance into any network that is not series-parallel.

Recall the Braess graph G_B, shown in Figure 2.

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[5] The inequality might be strict. Consider the case of 2 parallel links with $(\ell_1(x), \sigma_1(x)) = (1, x)$ and $(\ell_2(x), \sigma_2(x)) = (2, 0)$, and 1 unit of flow, half with $r = 0$ and half with $r = 2$.

Proposition 1. *For any strictly heterogeneous demand on the Braess graph G_B, there exist edge functions $(\ell_e)_{e \in E}$ and $(\sigma_e)_{e \in E}$ that depend on the demand, for which $C^{ht}(g) > C^{hm}(f)$. In addition, this remains true if we are restricted to only using affine functions.*

Proof. We may assume WLOG that the demand is of unit size.

Let $\bar{r}$ be the average diversity parameter and let $r_{\min}$ be the infimum of the diversity parameters' distribution. Let r_0 be any diversity parameter and let d_0 be the total demand with diversity parameter $\leq r_0$. Suppose that in addition, $r_{\min} \leq r_0 < \bar{r}$, and the corresponding d_0 satisfies $d_0 > 0$. As the demand is strictly heterogeneous, there must be such an r_0. Later on, r_0 will be specified further.

In addition, we let h be any continuous, strictly increasing cost function with $h(\frac{1}{2}) = 1$ and $h(\frac{1}{2} + \frac{d_0}{2}) = 1 + \frac{\bar{r} - r_0}{2}$.

Consider the Braess graph $G_B = (\{s, u, v, t\}, \{(s, u), (u, t), (u, v), (s, v), (v, t)\})$ with cost functions $\ell_{(s,u)}(x) = \ell_{(v,t)}(x) = h(x)$, $\sigma_{(s,u)}(x) = \sigma_{(v,t)}(x) = 0$, $\ell_{(u,t)}(x) = \ell_{(s,v)}(x) = 2 + \frac{\bar{r} + r_0}{2}$, and $\sigma_{(u,t)}(x) = \sigma_{(s,v)}(x) = 0$, and $\ell_{(u,v)}(x) = 1$ and $\sigma_{(u,v)}(x) = 1$. The instance is shown in Figure 2.
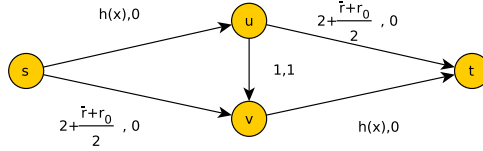

Fig. 2. The Braess network with the edge functions of Proposition 1. The pair $a(x), b(x)$ on each edge denotes the latency and deviation functions, respectively.

The heterogeneous equilibrium g routes d_0 units of flow through the zig-zag path, i.e. path s, u, v, t; the rest of the flow is split between the upper and lower paths s, u, t and s, v, t. This follows because with this routing, for players of diversity parameter $r \leq r_0$, the zig-zag path costs $2(1 + \frac{\bar{r} - r_0}{2}) + 1 + r \leq 2(1 + \frac{\bar{r} - r_0}{2}) + 1 + r_0 = 3 + \bar{r}$ while the other paths cost $1 + \frac{\bar{r} - r_0}{2} + 2 + \frac{\bar{r} + r_0}{2} = 3 + \bar{r}$, and for a player of diversity parameter $r \geq r_0$, the upper and lower paths cost $1 + \frac{\bar{r} - r_0}{2} + 2 + \frac{\bar{r} + r_0}{2} = 3 + \bar{r}$ while the zig-zag path costs $2(1 + \frac{\bar{r} - r_0}{2}) + 1 + r \geq 2(1 + \frac{\bar{r} - r_0}{2}) + 1 + r_0 = 3 + \bar{r}$.

To compute $C^{ht}(g)$, first note that players of diversity parameter $r > r_0$, who have total demand equal to $1 - d_0$, have cost $3 + \bar{r}$, and all players of any diversity parameter $r \leq r_0$ have cost $2(1 + \frac{\bar{r} - r_0}{2}) + 1 + r \geq 2(1 + \frac{\bar{r} - r_0}{2}) + 1 + r_{\min} = 3 + \bar{r} + r_{\min} - r_0$. The total cost of g is thus $C^{ht}(g) \geq d_0(3 + \bar{r} + r_{\min} - r_0) + (1 - d_0)(3 + \bar{r}) = 3 + \bar{r} + d_0(r_{\min} - r_0)$.

The homogeneous equilibrium f uses only the upper and lower paths. This follows because with this routing, for the average diversity parameter, the upper and lower paths cost $1 + 2 + \frac{\bar{r} + r_0}{2} = 3 + \frac{\bar{r} + r_0}{2}$, while the zig-zag path costs $1 + 1 + \bar{r} + 1 = 3 + \bar{r} > 3 + \frac{\bar{r} + r_0}{2}$. The total cost of f is thus $C^{hm}(f) = 3 + \frac{\bar{r} + r_0}{2}$.

Now we further specify r_0 so as to ensure $C^{ht}(g) > C^{hm}(f)$. By the above computations, it suffices to prove the existence of an r_0 that satisfies $r_{\min} \leq r_0 < \bar{r}$ and $d_0 > 0$ and in addition satisfies

$$3 + \bar{r} + d_0(r_{\min} - r_0) > 3 + \frac{\bar{r} + r_0}{2} \iff \frac{\bar{r} - r_0}{2} + d_0(r_{\min} - r_0) > 0.$$

As r_0 (which is $< \bar{r}$) goes to $r_{\min}$, the strictly positive quantity $\frac{\bar{r} - r_0}{2}$ increases and the non-positive quantity $d_0(r_{\min} - r_0)$ goes to 0 (because d_0 decreases and r_0 goes to $r_{\min}$). On the other hand, by definition, $r_{\min}$ is the infimum of the diversity parameters, and thus for any $\epsilon > 0$, there is a positive demand with diversity parameter $r \leq r_{\min} + \epsilon$. Therefore, there exists an r_0 satisfying the above inequality with $r_{\min} \leq r_0 < \bar{r}$ and $d_0 > 0$, as needed.

The above construction can be extended to only use affine functions. This can be done for example by changing function h to the linear function that satisfies $h(0) = 0$, $h(\frac{1}{2}) = A$ and $h(\frac{1}{2} + \frac{d_0}{2}) = A + \frac{\bar{r} - r_0}{2}$, and

for that A (in fact, $A = \frac{\bar{r}-r_0}{2d_0}$), only changing $\ell_{(s,v)}$, $\ell_{(u,t)}$ and $\ell_{(u,v)}$ to $\ell_{(s,v)}(x) = \ell_{(u,t)}(x) = 2A + \frac{\bar{r}+r_0}{2}$ and $\ell_{(u,v)}(x) = A$.

Theorem 2. *If G is not series-parallel, then for any strictly heterogeneous demand there are cost functions for which $C^{ht}(g) > C^{hm}(f)$.*

We defer the proof to the appendix. Instead, in the next section, we will enter into the more challenging construction needed for the multi-commodity case.

4   Topology of Multi-Commodity Networks for which Diversity Helps

In this section we fully characterize the topology of multi-commodity networks for which, with any choice of heterogeneous average-respecting demand and edge functions, diversity helps. Because of Theorem 2, if we require diversity to help on any instance on G, then for any commodity i, G_i needs to be series-parallel. Yet, as we shall see in Proposition 2, this is not enough. We also need to understand the overlaps of the G_i's. It turns out that the allowable overlaps are exactly captured by the topology of block-matching networks (Theorems 3 and 4). Ommited proofs can be found in Appendix A.

4.1   Sufficiency

Using Theorem 1, we can obtain an analogous theorem for the multi-commodity case.

Theorem 3. *Let G be a k-commodity block-matching network. Then, for any instance on G with average-respecting demand $C^{ht}(g) \leq C^{hm}(f)$.*

Proof. Consider Commodity i and let $G_i = s_i B_1 v_1 \dots v_{b_i-1} B_{b_i} t_i$ be its block representation. Consider an arbitrary B_j with terminals v_{j-1} and v_j. Because G is block-matching, any other Commodity l either contains B_j as a block in its block representation or contains none of its edges. Also, recall that, as explained in the preliminaries section, if G_l contains B_j, it has the same terminals v_{j-1} and v_j. This implies that under any routing of the demand, either all of l's demand goes through B_j or none of it does. This means that under both equilibria g and f, the total traffic routed from v_{j-1} to v_j through B_j is the same which further implies that, if restricted to the block, the cost of the heterogeneous equilibrium is less than or equal to that of the homogeneous equilibrium: $C^{ht}(g)\Big|_{B_j} \leq C^{hm}(f)\Big|_{B_j}$. For the latter, recall that the demand is average-respecting and thus f has a single average parameter. On the other hand, if we let $\mathcal{B}$ be the set of all the blocks of all commodities, then $C^{ht}(g) = \sum_{B \in \mathcal{B}} C^{ht}(g)\Big|_B$ and $C^{hm}(f) = \sum_{B \in \mathcal{B}} C^{hm}(f)\Big|_B$ which using the previous inequality proves the result. ∎

4.2   Necessity

To derive the necessity we first give an example of a non-block-matching network for which diversity hurts (Proposition 2). Then, after proving some properties for commodities for which the corresponding G_i are series-parallel (Lemmas 3 and 4), we mimic the above example to obtain contradicting instances for networks that are not block-matching and thereby prove Theorem 4.

Let G be the following 2-commodity network, depicted in Figure 3. G_2, the subnetwork for Commodity 2, consists of a simple s_2–t_2 path P_2, while G_1, the subnetwork for Commodity 1, is formed from two simple s_1–t_1 paths named P_1 and P_3; P_1 and P_2 are disjoint, while P_2 and P_3 share a single edge, named e_2. Finally e_1 is an edge on P_1 but not on P_3.

Proposition 2. *There exist edge functions and demands on G for which diversity hurts.*

Proof. Let $d_1 = d_2 = 1$ be the total demands for Commodities 1 and 2 respectively. Let G_1's demand consist of $\frac{3}{4}$ single-minded players and $\frac{1}{4}$ players with diversity parameter equal to 4, and let G_2's demand consist of players with diversity parameter equal to 1. To all edges other than e_1 and e_2, assign latency and deviation functions equal to 0. Assign edge e_1 the constant latency function $\ell_1(x) = 1$ and the constant deviation function $\sigma_1(x) = 2$. Assign edge e_2 the constant deviation $\sigma_2 = 0$, and as latency function any ℓ_2 that is continuous and strictly increasing, with $\ell_2(1) = 3$ and $\ell_2(\frac{5}{4}) = 9$.
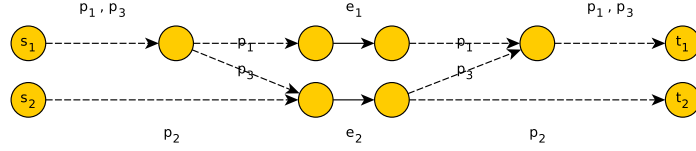

Fig. 3. The network for Proposition 2

The equilibrium costs depend only on the flow through edges e_1 and e_2, as all other edges have cost 0. Also note that at least 1 unit of flow will go through e_2 as this is the only route for G_2's demand.

In the heterogeneous equilibrium g of this instance, $\frac{3}{4}$ units of flow are routed through e_1, and $1 + \frac{1}{4}$ units of flow are routed through e_2, as then the single-minded players of G_1 compute a cost for P_1 equal to 1, and a cost for P_3 equal to 9, and thus prefer P_1, while the remaining players of G_1 compute a cost equal to 9 for both P_1 and P_3, and thus stay on P_3 (recall that ℓ_2 is strictly increasing). Consequently, the cost $C^{ht}(g)$ of the heterogeneous equilibrium is $C^{ht}(g) = 1 \cdot \frac{3}{4}d_1 + 9 \cdot \frac{1}{4}d_1 + 9 \cdot d_2 = 12$.

In the homogeneous equilibrium f, G_1's demand is all routed through e_1. This is because the average diversity parameter equals 1 and thus P_1 and P_3 are both computed to cost 3 (recall again that ℓ_2 is strictly increasing). Thus the cost $C^{hm}(f)$ of the homogeneous equilibrium is $C^{hm}(f) = 3 \cdot d_1 + 3 \cdot d_2 = 6$. Consequently, $C^{ht}(g) > C^{hm}(f)$, as needed.

Remark 1. The result would still hold if the common portion of P_1 and P_3 had a positive cost instead of zero cost. Again, it would still hold if the portion of P_2 after e_2 had a positive cost instead of zero cost. This is close to the way we will mimic this instance in the proof of Theorem 4. The idea, in both equilibria, is to route all the flow of Commodity 1 through two paths, P_1 and P_3, each containing one of e_1 or e_2, and to route the flow of Commodity 2 through a path, P_2, that contains e_2. This is done by putting (relatively) big constants as latency functions on all the edges that depart from vertices of the corresponding paths up to the point where e_1 or e_2 is reached, though some caution is needed. Then, the relation of the equilibria costs will follow as in Proposition 2, as the exact same edge functions will be used for edges e_1 and e_2. This will be specified precisely when we give the construction.

Next, we state some useful properties of series-parallel networks that are based on their block structure (the proofs are in the appendix). They will be used in the proof of Theorem 4.

Lemma 3. *Let i be a commodity of network G and suppose that G_i is series-parallel.*

(i) Let B_1 and B_2 be distinct blocks of G_i, with B_1 preceding B_2. There is no edge in G from an internal vertex of B_1 to an internal vertex of B_2.

(ii) Let u and v be vertices in G_i. If (u, v) is an edge of G then there is a simple s_i–t_i path in G_i that contains both u and v (not necessarily in that order).

Lemma 4. *Let i be a commodity of network G and suppose that G_i is series-parallel with block representation $G_i = s_i B_1 v_1 \ldots v_{b-1} B_b t_i$. Let w be a vertex of B_k for some $k \in [b]$.*

(i) Suppose that $w \neq v_{k-1}$, and let P be an arbitrary path from a vertex u, in a block A that precedes B_k in the block representation, to vertex w. Let w' be the first vertex on P that is an internal vertex in B_k, if any. Then P must include an edge of B_k exiting v_{k-1} prior to visiting w'.

(ii) Suppose that $w \neq v_k$. Then any path of G from w to a vertex u in a block succeeding B_k has to first enter v_k through one of its incoming edges that belong to B_k, before going to a block C that succeeds B_k in the block representation.

(iii) Every simple v_{k-1}–v_k path in G is completely contained in B_k.

Theorem 4. *Let G be a multi-commodity network. If diversity helps for every instance on G with average-respecting demand (i.e. for any heterogeneous equilibrium g and any homogeneous equilibrium f, $C^{ht}(g) \leq C^{hm}(f)$), then G is a block-matching network.*

Proof. Let G have k commodities. First, we note that for any $i \in [k]$, G_i is a series-parallel network. Otherwise, by Proposition 1, there is some heterogenous players' demand for Commodity i and edge functions for G_i such that diversity hurts. By letting all other commodities have zero demand we obtain an instance on G for which diversity hurts, a contradiction.

To prove that G is block-matching, it remains to show that for any two commodities i and j of G, for any block B of G_i and any block D of G_j, either $E(B) = E(D)$ or $E(B) \cap E(D) = \emptyset$. To reach a contradiction we assume otherwise, i.e. WLOG we assume that for Commodities 1 and 2 there exist two blocks B of G_1 and D of G_2 that share some common edge, and at the same time, WLOG, there is an edge in B that is not in D. The latter implies that B is not a single edge, and thus it must be a parallel combination of two series-parallel networks.

Let u and v be the endpoints of B. We first prove that all simple s_2–t_2 paths of G_2 that share an edge with B first traverse an edge starting at u before traversing any other edge of B (Proposition 3). Then we prove that all s_2–t_2 simple paths of G_2, that share an edge with B, reach u before traversing any internal vertex of B (Proposition 4). Since $E(B) \cap E(D) \neq \emptyset$, there is a simple s_2–t_2 path of G_2 that shares an edge with B. Proposition 4 implies that this path, Q, has a subpath consisting of a simple s_2–u path Q_1 that shares no internal vertex with B. A completely symmetric argument shows that Q has a subpath consisting of a simple v–t_2 path Q_3 that shares no internal vertex with B.[6] But then, for any simple u–v path Q_2 inside B, the path $Q' = Q_1, Q_2, Q_3$ is a simple s_2–t_2 path, and thus it belongs to G_2. But this implies that all the edges of B belong to G_2 and because B is a block, these edges will all be in a single block of G_2. This block must be block D, since by assumption $E(B) \cap E(D) \neq \emptyset$, contradicting the existence of an edge in B and not in D. Therefore, once these propositions are proved, the theorem will follow.

The proofs of these propositions rely on the same idea. For each proposition, assuming that it does not hold, we construct instances, i.e. we choose demand and edge functions for G, such that diversity hurts, contradicting the assumption that for any instance on G diversity helps. The construction of the contradicting instances is based on Remark 1, which follows Proposition 2.

Proposition 3. *Let P be a simple s_2–t_2 path in G_2 which shares an edge with B. The first edge on P in B departs from u, i.e. has the form (u, x) for some x in B.*

Proof. Let B be the parallel combination of H_1 and H_2. WLOG we may assume that P only visits vertices of G_1, plus s_2 and t_2, as we may treat subpaths of P that have vertices that lie outside G_1 as simple edges. Let w be the first internal vertex of P that belongs to B, and WLOG suppose that w lies in H_1. By Lemma 3(ii), the edge of P exiting w will either go toward t_1, i.e. forward, and thus traverse an edge of B for the first time (recall also Lemma 4(ii)), or will go toward s_1, i.e. backward, either staying in H_1 or going back to one of the preceding blocks of B. If it goes to one of the preceding blocks of B, then by Lemma 4(i), it has to

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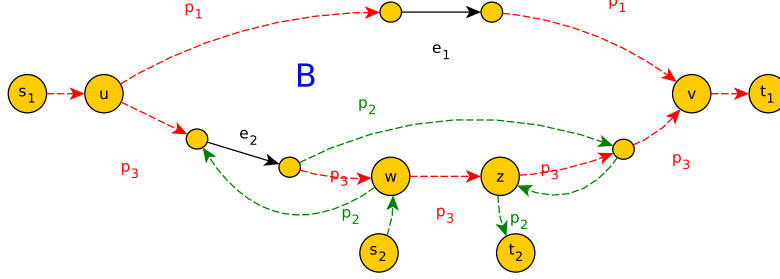[6] For the symmetric argument, simply reverse all the arcs and the directions of the demand.

Fig. 4. Illustrating why $P_2 \neq P$ in general in Proposition 3

traverse an edge of B departing from u in order to re-enter the internal portion of B (recall that P has some edge in B) and then the proposition would hold. The remaining possibility is that the backward edge leads to another internal vertex of H_1. However, we can only repeat this process finitely often so if the proposition does not hold, it must be that P eventually traverses a first edge in B that departs from an internal vertex of H_1. In this case we will reach a contradiction by creating an instance where diversity hurts. This instance will be based on the instance of Proposition 2.

We would like to use the following construction at this point. Let P_2 be the path P resulting from the discussion in the previous paragraph and let e_2 be the first edge on P that lies in B. Then let P_3 be an s_1–t_1 path through e_2. Recall that e_2 lies in H_1. Now let P_1 be an s_1–t_1 path that goes through H_2 and let e_1 be an arbitrary edge on P_1 in B. The intention is to force the s_1–t_1 flow to use just paths P_1 and P_3, while the s_2–t_2 flow uses just path P_2, at the same time ensuring that diversity is harmful as in Proposition 2. Consider the following edge functions. e_1 and e_2 receive the same edge functions as in Proposition 2. The other edges all receive a 0 deviation. For their latency functions, edges on P_1 and P_3 that are in B receive 0 functions. Outedges from P_1 and P_3 that lie in B all receive functions of constant value N even if they are on P_2. All as yet unassigned edges on P_2 receive 0 functions, and the remaining edges are all given functions of constant value $M \gg N$. However, the example in Figure A.1 shows that there is a zero cost s_2–t_2 path (s_2, w, z, t_2), which defeats the construction.

We fix this problem by defining the path P_2 as follows. Let x be the first vertex on path P (in the example, this is w) such that there is an edge (x, y) in B and such that there is a y–t_2 path P_{y-t_2} which does not go through any ealier vertex on P (i.e. any vertex from s_2 to x inclusive). Then (x, y) is chosen to be e_2, and P_2 is defined to be the simple path comprising the initial portion of P up to x, followed by e_2, followed by P_{y-t_2} (it may be that $P_2 = P$). Now the above cost functions, modulo a few details, will achieve the desired contradiction. These details follow.

Let path P_2 be the simple s_2–t_2 path that follows P up to w and keeps following it after w until for the first time it finds an edge of B (and hence of H_1) that departs from a vertex of P and can lead to t_2 via a path P' without returning to any of the previously visited vertices. Choose e_2 to be this edge and have P_2 follow e_2 and then path P' to t_2. Note that P itself can be P_2. Consider an arbitrary s_1–t_1 simple path P_3 that goes through e_2 (and thus through H_1) and an arbitrary s_1–t_1 simple path P_1 that goes through H_2, and let e_1 be any edge of P_1 in H_2. These paths and edges are going to mimic the ones of Proposition 2. By using some large values N and M as latency functions for some of the edges, at both equilibria, we will ensure that all the flow of G_1 goes through the subpaths of P_1 and P_3 in B, and all the flow of G_2 goes through the subpath of P_2 that starts at s_2 and ends with e_2.

Let $d_1 = d_2 = 1$ be the total demands for Commodities 1 and 2 respectively, and let all other commodities have 0 demand. Let G_1's demand consist of $\frac{3}{4}$ single-minded players and $\frac{1}{4}$ players with diversity parameter equal to 4, and let the demand of G_2 consist of players with diversity parameter equal to 1. Assign edge e_1 the constant latency function $\ell_1(x) = 1$ and the constant deviation function $\sigma_1(x) = 2$. Assign edge
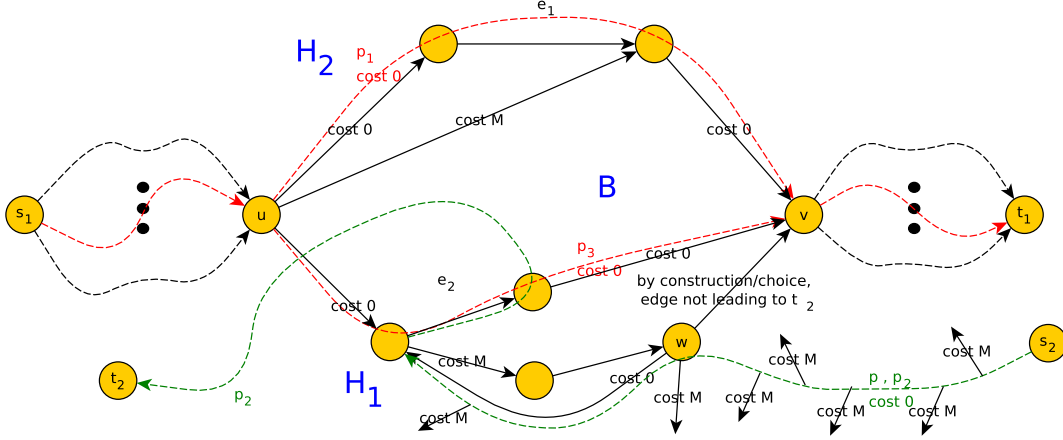
Fig. 5. Sample network for Proposition 3

e_2 the constant deviation $\sigma_2 = 0$, and as latency function any ℓ_2 that is continuous and strictly increasing with $\ell_2(1) = 3$ and $\ell_2(\frac{5}{4}) = 9$. Assign all other edges the constant deviation $\sigma_2 = 0$. Assign to all other edges of P_1 and P_3 that lie inside B 0 latency functions. To all edges that depart from a vertex of P_1 or P_3 and that lie on P_2, assign latency functions equal to some big constant N, say $N = 24$ (i.e. double the heterogeneous equilibrium cost of Proposition 2). For all remaining edges on P_2, assign 0 latency functions. Finally, to all remaining edges, assign constant latency functions equal to M, where M is defined to be $2|V(G)| \cdot N$.

Note that all edges other than e_2 have constant edge functions. Thus for Commodity 1 under both equilibria there will be a common cost C^{B^-} that will be paid on blocks other than B. Futhermore, for Commodity 2, any path that costs less than M will follow path P_2 up to and including e_2, for any edge leaving path P_2 either has cost M or is an edge of P_3 and by construction these edges do not create a cycle-free path to t_2. Thus if any of the homogeneous or heterogeneous equilibria is to cost less than M then all the flow of Commodity 2 will go through e_2, and from there follow a least cost path to t_2 (recall all edges other than e_2 have constant edge functions) with cost $C^{e_2^-}$ say. Note that $C^{e_2^-}$ will not be more than the cost of path P_2 following edge e_2, which is bounded by $|V(G)|N = \frac{M}{2}$, and thus the path portion with cost $C^{e_2^-}$ is preferable to any path with an edge of cost M.[7]

For the heterogeneous equilibrium g of this instance, for Commodity 1 route all the flow through the shortest s_1–u and v–t_1 paths, and inside B route $\frac{3}{4}$ units of flow through P_1 and $\frac{1}{4}$ units of flow through P_2, and route all the flow of commodity 2 through e_2, via P_2 up to e_2, and after e_2 via a least cost path to t_2. The single-minded players of G_1 compute a cost for P_1 equal to $1 + C^{B^-}$ and a cost for P_3 equal to $9 + C^{B^-}$, and thus prefer P_1 to P_3, while the remaining players of G_1 compute a cost equal to $9 + C^{B^-}$ for both P_1 and P_3, and thus prefer P_3 to P_1 (recall that ℓ_2 is strictly increasing). The other paths have cost at least $N + C^{B^-} = 24 + C^{B^-}$ and thus are not preferred by any type of players. The players of Commodity 2 pay cost equal to $9 + C^{e_2^-}$ (the cost of e_2 plus the cost after it) and thus prefer staying on e_2 rather than paying at least M to avoid it (recall $C^{e_2^-} \leq \frac{M}{2}$). Note that on P_2 for the vertices before e_2 there might be edges leaving P_2 that cost 0 (these are edges of P_3) but by the definition of P_2 they cannot lead to t_2 without visiting preceding vertices. Putting it all together, this routing is indeed the heterogeneous equilibrium with cost $C^{ht}(g) = 1 \cdot \frac{3}{4} d_1 + 9 \cdot \frac{1}{4} d_1 + d_1 C^{B^-} + 9 d_2 + d_2 C^{e_2^-} = 12 + C^{B^-} + C^{e_2^-}$.

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[7] For edge e_2 this will hold at both equilibria but in any case we can define its latency functions so that this holds in general, e.g. also set $\ell_2(2) = N$.

For the homogeneous equilibrium f of this instance, route all the demand of G_1 through e_1, via P_1 and the shortest s_1–u and v–t_1 paths, and route all the flow of Commodity 2 through e_2, via P_2 up to e_2 and from there via a least cost path to t_2. The average diversity parameter for Commodity 1's demand equals 1, and thus P_1 and P_3 are both computed to cost $3 + C^{B^-}$ while all other paths cost at least $N + C^{B^-} = 24 + C^{B^-}$ and thus are avoided. In the same way as above, the Commodity 2 players pay cost equal to $3 + C^{e_2^-}$ (the cost of e_2 plus the cost of the path portion following e_2) and thus prefer staying on e_2 rather than paying at least M to avoid it (recall $C^{e_2^-} \leq \frac{M}{2}$). Thus the cost $C^{hm}(f)$ of the homogeneous equilibrium is $C^{hm}(f) = 3 \cdot d_1 + d_1 \cdot C^{B^-} + 3 \cdot d_2 + d_2 \cdot C^{e_2^-} = 6 + C^{B^-} + C^{e_2^-}$. Consequently, $C^{ht}(g) > C^{hm}(f)$, as needed.

The above instance contradicts the assumption that G has the property that diversity helps for all edge functions. Thus the hypothesis that P, after reaching w (and possibly moving backward while staying in the internal portion of H_1) first traverses an edge of H_1 (departing from an internal vertex) does not hold. This was what needed for the proposition to hold.

Proposition 4. *All simple s_2–t_2 paths of G_2 that share an edge with B reach u before any internal vertex of B.*

Proof. Consider an arbitrary simple s_2–t_2 path P that shares some edge with B. WLOG we may assume, again, that P only visits vertices of G_1 and s_2, t_2, as we may handle subpaths of P that have vertices that lie outside G_1 as edges. Let B be the parallel combination of H_1 and H_2 and, assuming that the proposition does not hold, let w be the first vertex of P (before it reaches u) that belongs to the internal portion of B, and suppose WLOG that w lies in H_1. By Lemma 3(ii) and Proposition 3, the edge of P exiting w cannot go toward t_1, i.e. forward, as then it would traverse an edge of B for the first time that does not depart from u (recall also Lemma 4(ii)). Thus, it has to go toward s_1, i.e. backward, either staying in H_1 or going back to one of the preceding blocks of B. If by going backward it stays in H_1 then by the same argument it has again to move backward. However, we can only repeat this process finitely often and eventually after possibly hitting some vertices of H_1 other than w and after possibly visiting vertices of blocks that precede B, P hits u. Note that this happens without P having hit any vertex of H_2 prior to hitting u (recall Lemma 4(i)). There are two cases.

The first case occurs when on traversing P up to u, there is some edge of B departing from u that does not lead to t_2 without revisiting one of the previously visited vertices. Yet, since P shares an edge with B, by Proposition 3 there is an edge of B that departs from u and leads to t_2 without revisiting one of the previously visited vertices. Let e_1 be the first of the above edges and e_2 be the second one. For these edges, the same contradicting instance as in Proposition 3 can be constructed. Path P_1 is an arbitrary simple s_1–t_1 path containing e_1, path P_3 is an arbitrary simple s_1–t_1 path containing e_2, and path P_2 is constructed by following P up to u (instead of some internal vertex of H_1 from which e_2 was departing) and from there taking e_2 and then a path that leads to t_2 without revisiting vertices. The edge function assignment will be exactly the same. Diversity hurting, and thus the contradiction, will follow in the same way as in Proposition 3.

The second and more interesting case occurs when on following P up to u, all edges of B departing from u can lead to t_2 without revisiting one of the previously visited vertices. Let e_2 be an edge of H_2 (departing from u) with this property, let P_2 be the simple path that follows P up to u and then follows some path through e_2 to go to t_2, and let P_3 be a simple s_1–t_1 path that follows an arbitrary s_1–u path and an arbitrary v–s_2 path, and between u and v follows a path that contains e_2. Let P_1 be any path that follows an arbitrary s_1–u path and an arbitrary v–t_2 path, and between u and v follows a path that contains w, and therefore goes through H_1. Let e_1 be the edge of P_1 that departs from w. Note that, because of Proposition 3, e_1 cannot lead to t_2 with a simple path, i.e. without visiting vertices on P before w. This is a key fact for the contradiction to come and relates to the extra caution needed for this proof in comparison with the proof of

Proposition 3. See Figure 6 for a high level description of the instance. The rest of the details can be found in the appendix, Section A.6.

Acknowledgments

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APPENDIX

A   Ommited proofs

A.1   An instance without average-respecting demand where diversity hurts
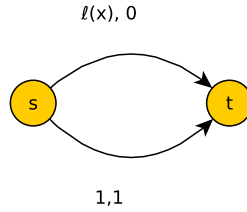
Let ℓ be any strictly increasing function with $\ell(1) = 2$ and $\ell(10/9) = 10$. Let there be two commodities, each of unit demand. The first commodity has demand that consists of homogeneous players with parameter equal to 10. The second commodity has demand that consists of 1/9 players with parameter equal to 9 and $8/9$ players with parameter equal to 0. The heterogeneous equilibrium has cost equal to 12 (only single-minded players are routed through the lower edge) while the homogeneous equilibrium has cost equal to 4 (the first commodity uses only the upper edge and the second uses only the lower edge).

A.2   The proof of Lemma 1

The proof is by induction on the decomposition of the series-parallel network. The base case of G being a single edge e is trivial as $x_e = d_1 \geq d_2 = y_e$.

For the inductive step, first suppose that G is a series combination of two series-parallel networks G_1 and G_2. For $i = 1, 2$, let x^i be the restriction of flow x to G^i, and y^i the restriction of flow y. By the inductive hypothesis, there is an s–t path P_i in G_i such that for all $e \in P_i, x_e^i \geq y_e^i$. It suffices to set P to be the concatenation of P_1 with P_2.

Now assume that G is a parallel combination of two series-parallel networks G_1 and G_2. Again, for $i = 1, 2$, let x^i be the restriction of flow x to G^i, and y^i of flow y. We may assume WLOG that the flow d_1^1 that G_1 receives in x^1 is at least as large as the flow d_2^1 that it receives in y^1, and further that $d_1^1 > 0$. By the inductive hypothesis applied to G_1 with demands d_1^1 and d_2^1, we obtain that there exists an s–t path P such that for all $e \in P, x_e^1 \geq y_e^1$ and this implies that for all $e \in P, x_e \geq y_e$, as needed.

A.3   The proof of Theorem 2

If G is not series-parallel then the Braess graph can be embedded in it (see e.g. Milchtaich [2006] or Valdes *et al.* [1979]). Thus, starting from the Braess network G_B, by subdividing edges, adding edges and extending one of the terminals by one edge, we can obtain G. Fix such a sequence of operations. For the given heterogeneous demand, we start from the Braess instance given by Proposition 1 and apply the sequence of operations one by one. Each time an edge addition occurs, we give the new edge a constant latency function equal to some large M and deviation equal to 0, each time an extension of the terminal occurs we give the

new edge a constant latency and deviation equal to 0, while each time an edge division occurs, if it is an edge with latency function M we give both edges latency function equal to M and deviation equal to 0, otherwise we give one of the two edges the latency and the deviation functions of the edge that got divided and we give the other one a constant latency and deviation equal to 0.

It is not hard to see that taking $M = 2(3 + \bar{r})$, i.e. more than double the heterogeneous cost of the instance of Proposition 1, (or $M = 2(3A + \bar{r})$ if we must only use affine functions), suffices to ensure that all edges having latency M and deviation 0 receive zero flow in both the heterogeneous equilibrium g and the homogeneous equilibrium f. In more detail, the only s–t routes that may have cost $< M$ are those that starting from s reach, with zero cost, some s' that corresponds to the s of the instance of Proposition 1, follow some path that corresponds to one of the upper, zig-zag, or lower paths of the instance of Proposition 1, with corresponding cost, reach some t' that corresponds to t of the instance of Proposition 1 and from there reach t with zero cost. This can be formally proved by induction on the number of embedding steps. Thus, $C^{ht}(g) > C^{hm}(f)$ can be derived in the same way as in Proposition 1 and that is enough to prove the theorem.

A.4   The proof of Lemma 3

For (i), if there were such an edge then a simple s_i–t_i path would be created that avoids the separators that lie between B_1 and B_2, contradicting the definition of G_i's block structure.

For (ii), let u and v be two vertices in G_i such that there is no simple s_i–t_i path in G_i that contains both of them. This implies that there is no edge between them in G_i. Now, in the series-parallel decomposition of G_i, let B be the smallest series-parallel subnetwork containing both u and v. By the choice of being smallest and the fact that there is no edge in G_i between u and v, B must be a composition of a B_1 containing u and a B_2 containing v. B_1 and B_2 are not connected in series because then there would be an s_i–t_i path in G_i containing both u and v. Therefore, B_1 and B_2 are connected in parallel; thus there cannot be any edge in G between u and v or else it would belong in some simple s_i–t_i path and therefore belong to G_i, violating G_i's series-parallel structure.

A.5   The proof of Lemma 4

The proofs of (i) and (ii) are by induction on the length of path u–w.

For (i), if the path has length equal to 1 then it is a simple edge, i.e. edge (u, w), and thus $u = v_{k-1}$, because of Lemma 3(i), and then (i) holds.

Now suppose inductively that the result holds for paths of length up to $l - 1$. Let u–w be a path of length l. Let (u, x) be the first edge on this path. Note that by Lemma 3(i), x cannot belong to any successor of B_k (unless $u = v_{k-1}$ and $x = v_k$, in which case (i) would hold). If x belongs to B_k and is not v_{k-1}, then by Lemma 3(i), $u = v_{k-1}$ and thus (i) holds. If $x = v_{k-1}$ or x does not belong to B_k (which implies it belongs to a predecessor of B_k) then the inductive hypothesis holds for the length $l - 1$ path x–w, yielding the desired edge exiting v_{k-1}. Thus (i) also holds for path u–w.

For (ii), if the path has length equal to 1 then it is a single edge, i.e. edge (w, u), and thus by Lemma 3(i)$u = v_k$, and (ii) holds. Now suppose inductively that the result holds for paths of length up to $l - 1$. Let w–u be a path of length l. Let (w, x) be the first edge on this path. Either $x = v_k$ or x is an internal vertex in B_k, by Lemma 3(i). If $x = v_k$ then (ii) holds. If x is an internal vertex in B_k, then the inductive hypothesis applies t the length $l - 1$ path x–w. Thus t(ii) also holds for path u–w.

(iii) follows from (i) and (ii). Consider an arbitrary simple v_{k-1}–v_k path P. Path P does not contain a vertex from any preceding block, for if it did, then to re-enter B_k so as to reach w, according to (i), it would go through v_{k-1} again, and then it would not be a simple path. Also, aside its endpoints, P does not contain a vertex from any succeeding block, for if it did, then to leave B_k, according to (ii), it would reach v_k before

reaching it again at the end, and then it would not be a simple path. Thus, a path that follows some simple s_i–v_{k-1} path, then follows P and then follows some simple v_k–t_i path is a simple s_i–t_i path. Consequently, P lies entirely in B_k.

A.6 The complete proof of Proposition 4

Consider an arbitrary simple s_2–t_2 path P that shares some edge with B. WLOG we may assume, again, that P only visits vertices of G_1 and s_2, t_2, as we may handle subpaths of P that have vertices that lie outside G_1 as edges. Let B be the parallel combination of H_1 and H_2 and, assuming that the proposition does not hold, let w be the first vertex of P (before it reaches u) that belongs to the internal portion of B, and suppose WLOG that w lies in H_1. By Lemma 3(ii) and Proposition 3, the edge of P exiting w cannot go toward t_1, i.e. forward, as then it would traverse an edge of B for the first time that does not depart form u (recall also Lemma 4(ii)). Thus, it has to go toward s_1, i.e. backward, either staying in H_1 or going back to one of the preceding blocks of B. If by going backward it stays in H_1 then by the same argument it has again to move backward. However, we can only repeat this process finitely often and eventually after possibly hitting some vertices of H_1 other than w and after possibly visiting vertices of blocks that precede B, P hits u. Note that this happens without P having hit any vertex of H_2 prior to hitting u (recall Lemma 4(i)). There are two cases.

The first case occurs when on traversing P up to u, there is some edge of B departing from u that does not lead to t_2 without revisiting one of the previously visited vertices. Yet, since P shares an edge with B, by Proposition 3 there is an edge of B that departs from u and leads to t_2 without revisiting one of the previously visited vertices. Let e_1 be the first of the above edges and e_2 be the second one. For these edges, the same contradicting instance as in Proposition 3 can be constructed. Path P_1 is an arbitrary simple s_1–t_1 path containing e_1, path P_3 is an arbitrary simple s_1–t_1 path containing e_2, and path P_2 is constructed by following P up to u (instead of some internal vertex of H_1 from which e_2 was departing) and from there taking e_2 and then a path that leads to t_2 without revisiting vertices. The edge function assignment will be exactly the same. Diversity hurting, and thus the contradiction, will follow in the same way as in Proposition 3.

The second and more interesting case occurs when on following P up to u, all edges of B departing from u can lead to t_2 without revisiting one of the previously visited vertices. Let e_2 be an edge of H_2 (departing from u) with this property, let P_2 be the simple path that follows P up to u and then follows some path through e_2 to go to t_2, and let P_3 be a simple s_1–t_1 path that follows an arbitrary s_1–u path and an arbitrary v–s_2 path, and between u and v follows a path that contains e_2. Let P_1 be any path that follows an arbitrary s_1–u path and an arbitrary v–t_2 path, and between u and v follows a path that contains w, and therefore goes through H_1. Let e_1 be the edge of P_1 that departs from w. Note that, because of Proposition 3, e_1 cannot lead to t_2 with a simple path, i.e. without visiting vertices on P before w. This is a key fact for the contradiction to come. See Figure 6.

To create the instance we proceed as in Proposition 3. Let $d_1 = d_2 = 1$ be the total demands for Commodities 1 and 2 respectively and let all other commodities have 0 demand. Let G_1's demand consist of $\frac{3}{4}$ single-minded players and $\frac{1}{4}$ players with diversity parameter equal to 4, and let the demand of G_2 consist of players with diversity parameter equal to 1. Assign edge e_1 the constant latency function $\ell_1(x) = 1$ and the constant deviation function $\sigma_1(x) = 2$. Assign edge e_2 the constant deviation $\sigma_2 = 0$, and as latency function any ℓ_2 that is continuous and strictly increasing with $\ell_2(1) = 3$ and $\ell_2(\frac{5}{4}) = 9$. Assign all other edges the constant deviation $\sigma_2 = 0$. To all other edges of P_1 and P_3 that lie inside B assign 0 latency functions. To all edges that depart from a vertex of P_1 or P_3 that lies on P_2 assign latency functions equal to some big constant N, say $N = 24$ (i.e. double the heterogeneous equilibrium cost of Proposition 2). For all remaining edges on P_2, assign 0 latency functions. Finally, to all remaining edges, assign constant latency functions equal to M, where M is defined to be $2|V(G)| \cdot N$.
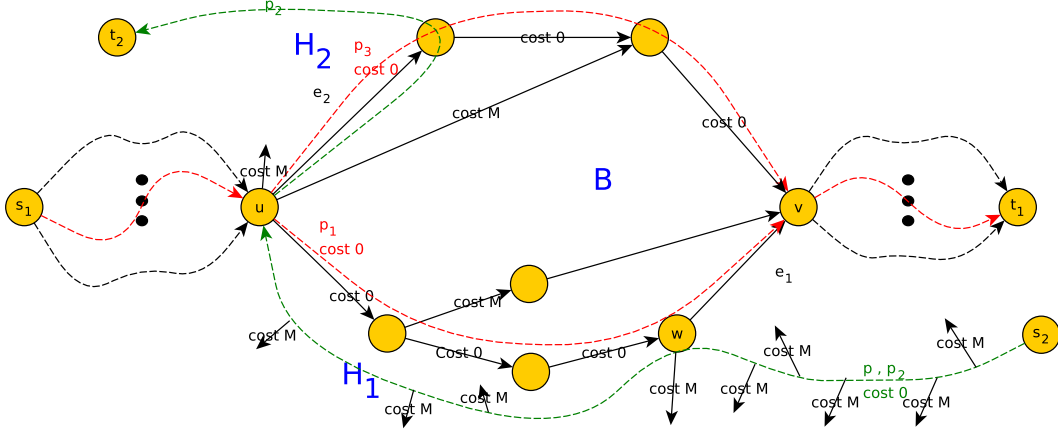
Fig. 6. Sample network for Proposition 4

Note that (as in Proposition 3) all edges other than e_2 have constant edge functions. Thus for both equilibria, Commodity 1 will have a common cost C^{B^-} that will be paid on blocks other than B. Also, for Commodity 2, any path that costs less than M will follow path P_2 up to and including e_2. For this, it suffices to show that all edges departing from vertices of P_1 in between u and w have cost M, as then if some portion of the flow, after visiting u, deviates and follows P_1 instead of e_2, then in order to avoid edges of cost M it will reach w which (as mentioned earlier) would be a dead end because of Proposition 3. This is proved in the next paragraph. Given that, if both the homogeneous and the heterogeneous equilibria are to cost less than M then all of Commodity 2's flow will go through e_2 and from there follow a shortest path to t_2 (recall that all other edges have constant edge functions) of cost $C^{e_2^-}$ say. Now note that $C^{e_2^-}$ will not be more than the cost of path P_2 following edge e_2, which is bounded by $|V(G)|N = \frac{M}{2}$, and thus the path portion with cost $C^{e_2^-}$ is preferable to any path with an edge of cost M.

Let E_M be the set of edges that depart from vertices of P_1 in between u and w. We want to prove that edges in E_M have cost M. By proposition 3, the vertices of P that belong to P_1 in between w and u, w included, have no departing edge that belongs to B and leads to t_2 without traversing preceding vertices of P. This implies that P and any other simple s_2–t_2 path that follows P up to u, cannot have some simple u–v path of H_2 as a subpath — call this Property X — otherwise, by letting P' be such a path, following P up to w, then picking any path inside H_1 that leads to v, and from there reaching t_2 via P', creates a simple s_2–t_2 path that has its first edge in B departing from an internal vertex, thereby contradicting Proposition 3. But if P_2 is to contain some edge in E_M then it has to leave H_2 and go to H_1. The only way to do that and keep its simplicity, because of Lemmas 3(ii) and 4(i), is by going to a block that succeeds B and then coming back to H_1. But by Lemma 4(ii), going to a block that succeeds B requires going through v first. Thus P_2 would have a complete u–v subpath that does not visit any other block, which by Lemma 4(iii) belongs in B and thus in H_2, contradicting Property X. Thus P_2 does not contain any edge in E_M.

Now, we compute the costs of the equilibria The heterogeneous equilibrium g, for Commodity 1, routes all the flow through the shortest s_1–u and v–t_1 paths, inside B routes $\frac{3}{4}$ units of flow through P_1, and $\frac{1}{4}$ units of flow through P_2, and routes all the flow of Commodity 2 through e_2, via P_2 up to e_2 and after e_2 via the shortest path to t_2. The single-minded players of G_1 compute a cost for P_1 equal to $1 + C^{B^-}$ and a cost for P_3 equal to $9 + C^{B^-}$, and thus prefer P_1 to P_3, while the remaining players of G_1 compute a cost equal to $9 + C^{B^-}$ for both P_1 and P_3, and thus prefer P_3 to P_1 (recall that ℓ_2 is strictly increasing). The other paths have cost at least $N + C^{B^-} = 24 + C^{B^-}$ and thus are not preferred by any type of player. The players of Commodity 2 pay cost equal to $9 + C^{e_2^-}$ (the cost of e_2 plus the cost after it) and thus prefer staying on e_2

rather than paying at least M to avoid e_2 (recall $C^{e_2^-} \leq \frac{M}{2}$). Also recall that on P_2, for the vertices before e_2, there might be edges leaving P_2 that have cost 0 (these are edges of P_1), but because of Proposition 3 they cannot lead to t_2 without visiting preceding vertices. Putting it all together this routing is indeed the heterogeneous equilibrium with cost $C^{ht}(g) = 1 \cdot \frac{3}{4}d_1 + 9 \cdot \frac{1}{4}d_1 + d_1 C^{B^-} + 9d_2 + d_2 C^{e_2^-} = 12 + C^{B^-} + C^{e_2^-}$.

The homogeneous equilibrium f routes all the demand of G_1 through e_1, via P_1 and the shortest s_1–u and v–t_1 paths, and routes all the flow of Commodity 2 through e_2, via P_2 up to e_2, and after e_2 via the shortest path to t_2. The average diversity parameter for Commodity 1's demand equals 1, and thus P_1 and P_3 are both computed to cost $3 + C^{B^-}$, while all other paths cost at least $N + C^{B^-} = 24 + C^{B^-}$ and thus are avoided. In the same way as above, the players of Commodity 2 pay cost equal to $3 + C^{e_2^-}$ (the cost of e_2 plus the cost after it) and thus prefer staying on e_2 rather than paying at least M to avoid it. Thus the cost $C^{hm}(f)$ of the homogeneous equilibrium is $C^{hm}(f) = 3 \cdot d_1 + d_1 \cdot C^{B^-} + 3 \cdot d_2 + d_2 \cdot C^{e_2^-} = 6 + C^{B^-} + C^{e_2^-}$. Consequently, $C^{ht}(g) > C^{hm}(f)$, contradicting the assumption that G satisfies that under any demand and edge functions diversity helps. Consequently, the proposition holds. $\qquad\square$